

RENORMALIZATION FOR PIECEWISE SMOOTH HOMEOMORPHISMS ON THE CIRCLE

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ABSTRACT. In this work we study the renormalization operator acting on piecewise smooth homeomorphisms on the circle, that turns out to be essentially the study of Rauzy-Veech renormalizations of generalized interval exchanges maps with genus one. In particular we show that renormalizations of such maps with zero mean nonlinearity and satisfying certain smoothness and combinatorial assumptions converges to the set of piecewise affine interval exchange maps.

CONTENTS

1. Introduction and results	1
1.1. Renormalization: Rauzy-Veech induction	3
1.2. Main results	6
2. Comparing compositions of $C^{2+\nu}$ maps with Moebius maps	7
3. Renormalization for genus one g.i.e.m.	14
3.1. Bounded Geometry for maps in $\mathcal{B}_k^{2+\nu}$	17
4. Symbolic representation	20
4.1. Admissible cylinders and its properties	21
References	29

1. INTRODUCTION AND RESULTS

One of the most studied classes of dynamical systems are orientation-preserving diffeomorphisms of the circle. It may be classified according

Date: August 10, 2011.

1991 *Mathematics Subject Classification.* 37E10, 37E05, 37E20, 37C05, 37B10.

Key words and phrases. renormalization, interval exchange transformations, Rauzy-Veech induction, universality, homeomorphism on the circle, convergence.

This work is based on the Ph. D. Thesis of the first author. K. C was partially supported by FAPESP 07/01045-7. D.S. was partially supported by FAPESP 2008/02841-4 and 2010/08654-1, CNPq 310964/2006-7 and 303669/2009-8.

to their rotation number $\rho(f)$ which, roughly speaking, measures the average rate of rotation of orbits around the circle. When $\rho(f) \in \mathbb{Q}$ then f has a periodic point and all other orbits will converge to some periodic orbit both in the future and in the past. If $\rho(f)$ is irrational then f has not periodic point and its dynamics depends on the smoothness of f . In this view point, Denjoy result ensures that if f is C^2 then it is conjugate to the rigid rotation of angle $\rho(f)$. In this context, a natural question to ask is under what condition the conjugacy is smooth. Several authors [KS87] [KO89], [SK89] have shown that if f is $C^{2+\nu}$ and $\rho(f)$ satisfies certain diophantine condition then the conjugacy will be at least C^1 .

A natural generalization of diffeomorphism of the circle are diffeomorphism with breaks, i.e., f has jumps in the first derivative on finitely many points. In [KV91] the authors show that for diffeomorphisms with one break the renormalization operator converges to a two dimensional space of the fractional linear transformation. Our first main result generalizes results in [KV91] for finitely many break points. A key combinatorial method in our proof is to consider a piecewise smooth homeomorphism on the circle as a generalized interval exchange transformation.

Let I be an interval and let $\mathcal{A}$ be a finite set (the *alphabet*) with $d \geq 2$ elements and $\mathcal{P} = \{I_\alpha : \alpha \in \mathcal{A}\}$ be an $\mathcal{A}$ -indexed partition of I into subintervals¹. We say that the triple $(f, \mathcal{A}, \mathcal{P})$, where $f : I \rightarrow I$ is a bijection, is a generalized interval exchange transformation with d intervals (g.i.e.m. with d intervals, for short), if $f|_{I_\alpha}$ is an orientation-preserving homeomorphism for each $\alpha \in \mathcal{A}$. Most of the time we will abuse the notation saying that f is a g.i.e.m. with d intervals. The order of the subintervals in the domain and image constitute the combinatorial data for f , where explicitly defined in the next section.

When $f|_{I_\alpha}$ is a translation we say which f is a standard i.e.m. Standard i.e.m. arise naturally as Poincaré return maps of measured foliations and geodesic flows on translation surfaces. But they are also interesting examples of simple dynamical systems with very rich dynamics and have been extensively studied for their own sake. When $d = 2$, by identifying the endpoints of I , standard i.e.m. correspond to rotations of the circle and generalized i.e.m. correspond to circle homeomorphisms.

¹All the subintervals will be bounded, close on the left and open on the right.

In [KS87] it is shown a new proof of M. Herman's theorem. From the viewpoint of the renormalization group ideology they show the convergence to the linear fixed point of the renormalization group transformation for diffeomorphisms of the circle. We use a similar approach to study generalized interval exchange maps of genus one.

1.1. Renormalization: Rauzy-Veech induction. To describe the combinatorial assumptions of our results, we need to introduce the Rauzy-Veech scheme. This is a renormalization scheme. Renormalization group techniques are a very powerful tool in one-dimensional dynamics. For example see [KV91] for circle homeomorphism and [dMvS93] for unimodal maps.

Following the algorithm of Rauzy [Rau79] and Veech [Vee78], for every i.e.m. f without connexions, we define the Rauzy-Veech induction by considering the first return maps f_n of f on a decreasing sequence of intervals I^n , with the same left endpoint than I . The map f_n is again generalized i.e.m. with the same alphabet $\mathcal{A}$ but the combinatorial data may be different.

Given two intervals J and U , we will write $J < U$ if their interior are disjoint and $x < y$ for every $x \in J$ and $y \in U$. This defines a partial order in the set of all intervals. Denote the length of an interval J by $|J|$.

Given a g.i.e.m. $f: I^0 \rightarrow I^0$, with alphabet $\mathcal{A}$. Let $\pi_j: \mathcal{A} \rightarrow \{1, \dots, d\}$, with $j = 0, 1$, be bijections such that

$$I_\alpha < I_\beta$$

iff $\pi_0(\alpha) < \pi_0(\beta)$ and

$$f(I_\alpha) < f(I_\beta)$$

iff $\pi_1(\alpha) < \pi_1(\beta)$.

The pair $\pi = \pi(f) = (\pi_0, \pi_1)$ is called the combinatorial data associated to the g.i.e.m. f . We call

$$(1) \quad p = \pi_1 \circ \pi_0^{-1}: \{1, \dots, d\} \rightarrow \{1, \dots, d\}$$

the *monodromy invariant* of the pair $\pi = (\pi_0, \pi_1)$. When appropriate we will use the notation $p = (p(1) \ p(2) \ \dots \ p(d))$ for the data combinatorial of f . We also assume that the pair $\pi = (\pi_0, \pi_1)$ is irreducible, i.e., for all $j \in \{1, \dots, d-1\}$ we have $\pi_1 \circ \pi_0^{-1}(\{1, \dots, j\}) \neq \{1, \dots, j\}$.

For each $\varepsilon \in \{0, 1\}$, define $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$. If $|I_{\alpha(0)}| \neq |f(I_{\alpha(1)})|$ we say that f is Rauzy-Veech renormalizable (or simply renormalizable, from now on). If $|I_{\alpha(0)}| > |f(I_{\alpha(1)})|$ we say that the letter $\alpha(0)$ is the winner and the letter $\alpha(1)$ is the loser. We say that f is type 0

renormalizable and we can define a map $\hat{R}(f)$ as the first return map of f to the interval

$$I^1 = I \setminus f(I_{\alpha(1)}).$$

Otherwise $|I_{\alpha(0)}| < |f(I_{\alpha(1)})|$, the letter $\alpha(1)$ is the winner and the letter $\alpha(0)$ is the loser, we say that f is type 1 renormalizable and we can define a map $\hat{R}(f)$ as the first return map of f to the interval

$$I^1 = I \setminus I_{\alpha(0)}.$$

We want to see $R(f)$ as a g.i.e.m. To this end we need to associate to this map a $\mathcal{A}$ -indexed partition of its domain. We do this in the following way. The subintervals of the $\mathcal{A}$ -partition $\mathcal{P}^1$ of I^1 are denoted by I_α^1 . If f has type 0, $I_\alpha^1 = I_\alpha$. If $\alpha \neq \alpha(0)$, $I_{\alpha(0)}^1 = I_{\alpha(0)} \setminus f(I_{\alpha(1)})$ and when f has type 1, $I_\alpha^1 = I_\alpha$ if $\alpha \neq \alpha(1), \alpha(0)$, $I_{\alpha(1)}^1 = f^{-1}(f(I_{\alpha(1)}) \setminus I_{\alpha(0)})$ and $I_{\alpha(0)}^1 = I_{\alpha(1)} \setminus I_{\alpha(1)}^1$. It is easy to see that in both cases (type 0 and 1) we have

$$R(f)(x) = \begin{cases} f^2(x), & \text{se } x \in I_{\alpha(1-\varepsilon)}^1, \\ f(x), & \text{otherwise.} \end{cases}$$

and $(R(f), \mathcal{A}, \mathcal{P}^1)$ is a g.i.e.m., called the **Rauzy-Veech renormalization** (or simply renormalization, for short) of f . If f is renormalizable with type $\varepsilon \in \{0, 1\}$ then the combinatorial data $\pi^1 = (\pi_0^1, \pi_1^1)$ of $R(f)$ is given by

$$\pi_\varepsilon^1 := \pi_\varepsilon$$

and

$$\text{and } \pi_{1-\varepsilon}^1(\alpha) = \begin{cases} \pi_{1-\varepsilon}(\alpha), & \text{se } \pi_{1-\varepsilon}(\alpha) \leq \pi_{1-\varepsilon}(\alpha(\varepsilon)), \\ \pi_{1-\varepsilon}(\alpha) + 1, & \text{se } \pi_{1-\varepsilon}(\alpha(\varepsilon)) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}(\alpha(\varepsilon)) + 1, & \text{se } \pi_{1-\varepsilon}(\alpha) = d. \end{cases}$$

Since π^1 depends only on π and the type ε , we denote $r_\varepsilon(\pi) = \pi^1$.

A g.i.e.m. is infinitely renormalizable if $R^n(f)$ is well defined, for every $n \in \mathbb{N}$. For every interval of the form $J = [a, b)$ we denote $\partial J = \{a\}$. We say that a g.i.e.m f has no connection if

$$f^m(\partial I_\alpha) \neq \partial I_\beta \text{ for all } m \geq 1, \alpha, \beta \in \mathcal{A} \text{ with } \pi_0(\beta) \neq 1.$$

Let ε_n the type of the n -th renormalization, $\alpha_n(\varepsilon_n)$ be the winner and $\alpha_n(1 - \varepsilon_n)$ be the loser of the n -th renormalization.

We say that infinitely renormalizable g.i.e.m. f has **k -bounded combinatorics** if for each n and $\beta, \gamma \in \mathcal{A}$ there exists $n_1, p \geq 0$,

with $|n - n_1| < k$ and $|n - n_1 - p| < k$, such that $\alpha_{n_1}(\varepsilon_{n_1}) = \beta$, $\alpha_{n_1+p}(1 - \varepsilon_{n_1+p}) = \gamma$ and

$$\alpha_{n_1+i}(1 - \varepsilon_{n_1+p}) = \alpha_{n_1+i+1}(\varepsilon_{n_1+i})$$

for every $0 \leq i < p$.

We say that a g.i.e.m. $f: I \rightarrow I$ has **genus one** [Vee82] (or belongs to the **rotation class** [NR97]) if f has at most two discontinuities. Note that every g.i.e.m. with either two or three intervals has genus one. If f is renormalizable and has genus one, it is easy to see that $R(f)$ has genus one.

Given two infinitely renormalizable g.i.e.m. f and g , defined with the same alphabet $\mathcal{A}$, we say that f and g have the **same combinatorics** if $\pi(f) = \pi(g)$ and the n -th renormalization of f and g have the same type, for every $n \in \mathbb{N}$. It follows that if $\pi^n(f) = \pi^n(g)$ for every n , where $\pi^n(f)$ is the combinatorial data of the n -th renormalization of f .

Definition 1.1. Let $\mathcal{B}_k^{2+\nu}$, $k \in \mathbb{N}$ and $\nu > 0$, be the set of g.i.e.m. $f: I \rightarrow I$ such that

- (i) For each $\alpha \in \mathcal{A}$ we can extend f to $\overline{I_\alpha}$ as an orientation-preserving diffeomorphism of class $C^{2+\nu}$;
- (ii) the g.i.e.m. f has k -bounded combinatorics;
- (iii) The map f has genus one and has no connection;

Let H be a non-degenerate interval, let $g: H \rightarrow \mathbb{R}$ be a diffeomorphism and let $J \subset H$ be an interval. We define the *Zoom* of g in H , denoted by $\mathcal{Z}_H(g)$, the transformation $\mathcal{Z}_H(g) = A_1 \circ g \circ A_2$, where A_1 and A_2 are orientation-preserving affine maps, which send $[0, 1]$ into H and $g(H)$ into $[0, 1]$ respectively. Consider the set $C^2([0, 1], \mathbb{R})$ of all C^2 functions $g: [0, 1] \rightarrow \mathbb{R}$ with the usual norm

$$d_{C^2}(f, g) := \sum_{i=0}^2 \sup_{x \in [0, 1]} |D^{(i)}f(x) - D^{(i)}g(x)|,$$

where $D^{(i)}f$ and $D^{(i)}g$ denote the i -th derivative of f and g respectively.

Denote by $\mathcal{M}$ the set of Möbius transformations $M: [0, 1] \rightarrow [0, 1]$ such that $M(0) = 0$ and $M(1) = 1$. Note that $\mathcal{M}$ is an one-dimensional real Lie group. Indeed any element $M \in \mathcal{M}$ has the form

$$(2) \quad M = M_N(x) = \frac{xe^{\frac{-N}{2}}}{1 + x(e^{\frac{-N}{2}} - 1)}.$$

for some $N \in \mathbb{R}$ and $M_{N_1} \circ M_{N_2} = M_{N_1+N_2}$. Moreover M_N is the unique Möebius transformation M which sends $[0, 1]$ onto $[0, 1]$, $M(0) = 0$, $M(1) = 1$, and

$$\int_0^1 \frac{D^2 M(x)}{DM(x)} dx = N.$$

1.2. Main results.

Theorem 1. *Let $f \in \mathcal{B}_k^{2+\nu}$. Then there are $C = C(f) > 0$ and $0 < \lambda = \lambda(k) < 1$ such that*

$$d_{C^2}(\mathcal{Z}_{I_\alpha^n}(R^n(f)), M_{N_\alpha^n}) \leq C\lambda^n$$

for all $\alpha \in \mathcal{A}$. Here

$$N_\alpha^n = \int_{I_\alpha^n} \frac{D^2 R^n(f)(x)}{DR^n(f)(x)} dx.$$

In particular

$$d_{C^2}(\mathcal{Z}_{I_\alpha^n}(R^n(f)), \mathcal{M}) \leq C\lambda^n.$$

We can say more about the mean nonlinearities N_α^n . Denote by $q_\alpha^n \in \mathbb{N}$ the first return time of the interval I_α^n to the interval I^n , i.e., $\hat{R}^n(f)|_{I_\alpha^n} = f^{q_\alpha^n}$, for some $q_\alpha^n \in \mathbb{N}$.

Theorem 2. *Let $f \in \mathcal{B}_k^{2+\nu}$. Then there are $C = C(f) > 0$ and $0 < \lambda = \lambda(k) < 1$ such that*

$$|N_\alpha^n - \frac{\sum_{i=1}^{q_\alpha^n} |f^i(I_\alpha^n)|}{|I|} \int \frac{D^2 f(x)}{Df(x)} dx| \leq C\lambda^{\sqrt{n}}.$$

In particular if

$$\int_{[0,1]} \frac{D^2 f(x)}{Df(x)} dx = 0$$

then $|N_\alpha^n| < C\lambda^{\sqrt{n}}$.

Our third result is an almost direct consequence of Theorem 1 and 2.

Theorem 3. *Let $f \in \mathcal{B}_k^{2+\nu}$ such that*

$$\int_{[0,1]} \frac{D^2 f(x)}{Df(x)} dx = 0.$$

Then there are $C = C(k) > 0$ and $0 < \lambda = \lambda(k) < 1$ such that

$$|\mathcal{Z}_{I_\alpha^n}(R^n(f)) - \text{Id}|_{C^2} \leq C \cdot \lambda^{\sqrt{n}} \text{ for all } \alpha \in \mathcal{A}.$$

The structure of this paper is as follows. In Section §2 we describe general results on compositions of diffeomorphisms of class $C^{2+\nu}$. In Section §3 we study renormalization of generalized interval exchange maps of genus one and prove Theorem 1. In Section §4 we codify the dynamics of f using an specially crafted symbolic dynamics to obtain finer geometric properties of the partitions associated with renormalizations of f and we finally prove Theorem 2 and Theorem 3.

This is the first of a series of two papers based on the Ph. D. Thesis of the first author [Cun10]. In the forthcoming second work we continue our study of the renormalization operator for generalized interval exchange transformations of genus one and its consequences, particularly the rigidity (universality) phenomena in the setting of piecewise smooth homeomorphisms on the circle.

2. COMPARING COMPOSITIONS OF $C^{2+\nu}$ MAPS WITH MOEBIUS MAPS

In this section, we show some results about composition of $C^{2+\nu}$ -diffeomorphisms, comparing these compositions with Moebius maps. Let $f : [a, b] \rightarrow [f(a), f(b)]$ be an C^2 orientation-preserving diffeomorphism. Define the *nonlinearity function* $n_f : [a, b] \rightarrow \mathbb{R}$ by

$$n_f(x) = \frac{D^2 f(x)}{Df(x)} = D(\ln Df(x)).$$

Notice that

$$n_{f \circ g}(x) = n_f(g(x))Dg(x) + n_g(x),$$

consequently if f_i are C^2 diffeomorphisms such that $f = f_n \circ \dots \circ f_1$ is defined in $[a, b]$ we have

$$(3) \quad \int_{[a,b]} \frac{D^2 f(x)}{Df(x)} dx = \sum_{i=1}^n \int_{f^{i-1}[a,b]} \frac{D^2 f_i(x)}{Df_i(x)} dx.$$

If $[a, b] = [0, 1]$ we define

$$N_f = \int_{[0,1]} \frac{D^2 f(x)}{Df(x)} dx.$$

The nonlinearity n_f defines f up its domain and image. Indeed, given a continuous function $n : [0, 1] \rightarrow \mathbb{R}$ there is unique C^2 -diffeomorphism $f : [0, 1] \rightarrow [0, 1]$ such that $f(0) = 0$, $f(1) = 1$ and $n_f = n$. Indeed [Mar98]

$$(4) \quad f(x) = \frac{\int_0^x \exp \left(\int_0^z n(y) dy \right) dz}{\int_0^1 \exp \left(\int_0^z n(y) dy \right) dz}.$$

Let $f: [0, 1] \rightarrow [f(0), f(1)]$ be an C^2 orientation-preserving diffeomorphism. If $[a, b] \subset [0, 1]$, let $\tilde{f} = \mathcal{Z}_{[a,b]}(f)$ be the Zoom of f in $[a, b]$. Then

$$(5) \quad n_{\mathcal{Z}(f)}(x) = (b - a) \cdot n_f(a + x(b - a)).$$

Suppose

$$(6) \quad |n_f(x) - n_f(y)| \leq C_0 \cdot |x - y|^\nu.$$

for $x, y \in [0, 1]$ and

$$|n_f|_{C^0[0,1]} := \sup_{x \in [0,1]} \{|n_f(x)|\} \leq C_1.$$

Then by (6) and (5) we have that

$$(7) \quad |n_{\mathcal{Z}(f)}(x) - n_{\mathcal{Z}(f)}(y)| \leq C_0 \cdot \delta^{1+\nu} \quad \text{and} \quad |n_{\mathcal{Z}(f)}|_{C^0[0,1]} \leq C_1 \cdot \delta,$$

with $x, y \in [0, 1]$ and $\delta = b - a$. Note that

$$(8) \quad N_{\mathcal{Z}(f)} = \int_0^1 (b - a) \cdot n_f(a + x(b - a)) \, dx = \int_{[a,b]} \frac{D^2 f(x)}{Df(x)} \, dx.$$

Proposition 2.1. *Let $f: [0, 1] \rightarrow [f(0), f(1)]$ be an orientation-preserving diffeomorphism of class $C^{2+\nu}$, $[a, b] \subset [0, 1]$ and define $\tilde{f} = \mathcal{Z}_{[a,b]}f$. Then*

$$d_{C^2}(\tilde{f}, M_{N_{\tilde{f}}}) = O(\delta^{1+\nu}),$$

where $\delta = b - a$.

Before we prove the Proposition 2.1 we prove the following lemma:

Lemma 2.2. *Let $N \in \mathbb{R}$. Let $f_N: [0, 1] \mapsto [0, 1]$ be a diffeomorphism such that $n_f(x) = N$ for all $x \in [0, 1]$, $f(0) = 0$, $f(1) = 1$. Then*

$$d_{C^2}(f_N, M_N) = O(N^2).$$

Proof. By (4) we have

$$f_N(x) = \frac{\int_0^x e^{Nz} dz}{\int_0^1 e^{Nz} dz} = \frac{e^{Nx} - 1}{e^N - 1}.$$

Therefore,

$$\begin{aligned}
|f_N(x) - M_N(x)| &= \left| \frac{e^{Nx} - 1}{e^N - 1} - \frac{xe^{\frac{-N}{2}}}{1 + x(e^{\frac{-N}{2}} - 1)} \right| \\
&= \left| \frac{Nx + \frac{N^2x^2}{2} + O(N^3)}{N + \frac{N^2}{2} + O(N^3)} - \frac{x(1 - \frac{N}{2} + O(N^2))}{1 + x(-\frac{N}{2} + O(N^2))} \right| \\
&= \left| x(1 + \frac{Nx}{2} - \frac{N}{2}) + O(N^2) - x(1 + \frac{Nx}{2} - \frac{N}{2}) + O(N^2) \right| \\
&= O(N^2).
\end{aligned}$$

$$\begin{aligned}
|Df_N(x) - DM_N(x)| &= \left| \frac{Ne^{Nx}}{e^N - 1} - \frac{e^{\frac{-N}{2}}}{[1 + x(e^{\frac{-N}{2}} - 1)]^2} \right| \\
&= \left| \frac{(1 + Nx + O(N^2))}{(1 + \frac{N}{2} + O(N^2))} - e^{-\frac{N}{2}}(1 + \frac{Nx}{2} + O(N^2))^2 \right| \\
&= \left| (1 + Nx)(1 - \frac{N}{2}) + O(N^2) - e^{-\frac{N}{2}}(1 + Nx + O(N^2)) \right| \\
&= \left| 1 - \frac{N}{2} + Nx + O(N^2) - 1 - \frac{N}{2} + Nx + O(N^2) \right| \\
&= O(N^2).
\end{aligned}$$

$$\begin{aligned}
|D^2f_N(x) - D^2M_N(x)| &= \left| \frac{N^2e^{Nx}}{e^N - 1} - \frac{-2e^{\frac{-N}{2}}(e^{-\frac{N}{2}} - 1)}{[1 + x(e^{\frac{-N}{2}} - 1)]^3} \right| \\
&= \left| N + O(N^2) - (N + O(N^2))(1 + \frac{Nx}{2} + O(N^2))^3 \right| \\
&= O(N^2).
\end{aligned}$$

□

Proof of Proposition 2.1. By Eq. (7) we have

$$|N_{\tilde{f}}| \leq |n_f|_{C^0[0,1]} \delta.$$

To simplify the notation, denote $\tilde{N} = N_{\tilde{f}}$. First note that

$$d_{C^2}(\tilde{f}, M_{\tilde{N}}) \leq d_{C^2}(\tilde{f}, f_{\tilde{N}}) + d_{C^2}(f_{\tilde{N}}, M_{\tilde{N}}).$$

In view of Lemma (2.2), only need to estimate the first term of the right hand side. For this note that $\tilde{N} = \int_0^1 n_{\tilde{f}}(s) ds = n_{\tilde{f}}(\theta)$ for some

$\theta \in [0, 1]$. If

$$|n_f(x) - n_f(y)| \leq C_0 |x - y|^\nu,$$

then by Eq. (7) we have $|n_{\tilde{f}}(x) - \tilde{N}| = |n_{\tilde{f}}(x) - n_{\tilde{f}}(\theta)| \leq C_0 \cdot \delta^{1+\nu}$, so $n_{\tilde{f}}(x) = \tilde{N} + O(\delta^{1+\nu})$. Then by (4) we obtain

$$(9) \quad \tilde{f}(x) = x(1 - \frac{\tilde{N}}{2} + \frac{\tilde{N}}{2}x + O(\delta^{1+\nu}));$$

$$(10) \quad D\tilde{f}(x) = 1 + \tilde{N}x - \frac{\tilde{N}}{2} + O(\delta^{1+\nu});$$

$$(11) \quad D^2\tilde{f}(x) = \tilde{N} + O(\delta^{1+\nu}).$$

Using the estimates for $f_{\tilde{N}}$, $Df_{\tilde{N}}$ and $D^2f_{\tilde{N}}$ similar to those in the proof of Lemma 2.2, the proof is complete. $\square$

From now on let $f_i : [0, 1] \rightarrow [0, 1]$, $i \in \mathbb{N}$ be orientation-preserving diffeomorphisms of class $C^{2+\nu}$, with $f_i(0) = 0$, $f_i(1) = 1$, and such that there exist $C_0, C_1 > 0$ satisfying

$$(12) \quad |n_{f_i}(x) - n_{f_i}(y)| \leq C_0 \cdot |x - y|^\nu.$$

and

$$|n_{f_i}|_{C^0[0,1]} \leq C_1.$$

for every $i \in \mathbb{N}$. Let $[a_i, b_i] \subset [0, 1]$, $\delta_i = b_i - a_i$, and $\tilde{f}_i = \mathcal{Z}_{[a_i, b_i]}(f_i)$, $M_i = M_{N_{\tilde{f}_i}}$,

$$\tilde{f}_1^n = \tilde{f}_n \circ \tilde{f}_{n-1} \circ \cdots \circ \tilde{f}_1 \quad \text{and} \quad M_1^n = M_n \circ M_{n-1} \circ \cdots \circ M_1.$$

The following Proposition is the main result of this section. It compares the compositions of $\tilde{f}_i$'s and M_i 's.

Proposition 2.3 (see also [KV91]). *Let f_i be as above. Then for every $C_2 > 0$ there exists $C_3 > 0$ with the following property. If $\sum_{i=1}^n \delta_i \leq C_2$, then*

$$|\tilde{f}_1^n - M_1^n|_{C^2} \leq C_3 \cdot (\max_{1 \leq j \leq n} \delta_j)^\nu.$$

The proof of Theorem 1 in [KV91] is the main motivation to Proposition 2.3. Since Proposition 2.3 is not stated explicitly in [KV91] in its full generality, we include the full argument for the sake of completeness. Before we prove this proposition, we need some lemmas.

Lemma 2.4. *There is $C_4 = C_4(C_1, C_2) > 0$ such that*

$$e^{-C_4} \leq D\tilde{f}_1^n(x) \leq e^{C_4},$$

for all $x \in [0, 1]$ and for all $n \geq 0$.

Proof.

$$\begin{aligned}
\ln \frac{D\tilde{f}_1^n(x)}{D\tilde{f}_1^n(y)} &= \ln \frac{D\tilde{f}_n(\tilde{f}_1^{n-1}(x)) \cdot D\tilde{f}_{n-1}(\tilde{f}_1^{n-2}(x)) \cdots D\tilde{f}_1(x)}{D\tilde{f}_n(\tilde{f}_1^{n-1}(y)) \cdot D\tilde{f}_{n-1}(\tilde{f}_1^{n-2}(y)) \cdots D\tilde{f}_1(y)} \\
&= \sum_{j=1}^n \ln D\tilde{f}_j(\tilde{f}_1^{j-1}(x)) - \ln D\tilde{f}_j(\tilde{f}_1^{j-1}(y)) \\
&= \sum_{j=1}^n \int_{\tilde{f}_1^{j-1}(y)}^{\tilde{f}_1^{j-1}(x)} \frac{D^2\tilde{f}_j(s)}{D\tilde{f}_j(s)} ds \\
&= \sum_{j=1}^n \frac{D^2\tilde{f}_j(z_{j-1})}{D\tilde{f}_j(z_{j-1})} |\tilde{f}_1^{j-1}(x) - \tilde{f}_1^{j-1}(y)|,
\end{aligned}$$

for some $z_{j-1} \in [\tilde{f}_1^{j-1}(y), \tilde{f}_1^{j-1}(x)]$.

Therefore by (7) we have

$$\left| \ln \frac{D\tilde{f}_1^n(x)}{D\tilde{f}_1^n(y)} \right| \leq \sum_{j=1}^n \left| \frac{D^2\tilde{f}_j(z_{j-1})}{D\tilde{f}_j(z_{j-1})} \right| \leq C_1 \cdot \sum_{j=1}^n \delta_j \leq C_1 C_2 = C_4.$$

Taking $y \in [0, 1]$ such that $D\tilde{f}_1^n(y) = 1$ we have the result. $\square$

Lemma 2.5. *There is $C_5 = C_5(C_1, C_2) > 0$ such that*

$$|D^2\tilde{f}_1^n(x)| \leq C_5,$$

for all $x \in [0, 1]$ and for all $n \geq 0$.

Proof. Note that

$$\begin{aligned}
|D^2\tilde{f}_1^n(x)| &= \left| \frac{D^2\tilde{f}_1^n(x)}{D\tilde{f}_1^n(x)} \right| \cdot |D\tilde{f}_1^n(x)| \\
&= \left| \sum_{j=1}^n \frac{D^2\tilde{f}_j(\tilde{f}_1^{j-1}(x))}{D\tilde{f}_j(\tilde{f}_1^{j-1}(x))} \cdot D\tilde{f}_1^{j-1}(x) \right| \cdot |D\tilde{f}_1^n(x)| \\
&\leq e^{2C_4} \sum_{j=1}^n \left| \frac{D^2\tilde{f}_j(\tilde{f}_1^{j-1}(x))}{D\tilde{f}_j(\tilde{f}_1^{j-1}(x))} \right| \\
&\leq e^{2C_4} \cdot C_1 \cdot \sum_{j=1}^n \delta_j \leq e^{2C_4} \cdot C_1 C_2 = C_5.
\end{aligned}$$

$\square$

Lemma 2.6. *There are $C_6 = C_6(C_1, C_2), C_7 = (C_1, C_2) > 0$ such that*

$$e^{-C_6} \leq |DM_1^n(x)| \leq e^{C_6}, \quad |D^2M_1^n(x)| \leq C_7, \quad |D^3M_1^n(x)| \leq C_8.$$

for all $x \in [0, 1]$ and for all $n \geq 0$.

Proof. Since $\mathcal{M}$ is a commutative Lie group, we have $M_1^n = M_N$, where $N = \sum_{i=1}^n N_{\tilde{f}_i}$. By Eq. (7) we have $|N_{\tilde{f}_i}| < C_1 \delta_i$, so

$$|N| \leq C_1 \cdot C_2.$$

One can easily use Eq. (2) to obtain estimates for $D^i M_1^n$, $i = 1, 2, 3$. $\square$

Proof of Proposition 2.3. We write

$$\tilde{f}_1^n - M_1^n = \sum_{i=1}^n M_{i+1}^n \circ \tilde{f}_1^i - M_i^n \circ \tilde{f}_1^{i-1},$$

where $M_{n+1}^n = f_1^0 = \text{Id}$. Then

$$\begin{aligned} |\tilde{f}_1^n(x) - M_1^n(x)| &\leq \sum_{i=1}^n \left| M_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1})(x) - M_{i+1}^n(M_i \circ \tilde{f}_1^{i-1})(x) \right| \\ &\leq e^{C_6} \cdot \sum_{i=1}^n \left| \tilde{f}_i \circ \tilde{f}_1^{i-1}(x) - M_i \circ \tilde{f}_1^{i-1}(x) \right| \\ &\leq e^{C_6} \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} \\ &\leq e^{C_6} \cdot C \cdot \left(\max_{1 \leq i \leq n} \delta_i \right)^\nu \sum_i \delta_i \\ &\leq e^{C_6} \cdot C \cdot C_2 \cdot \left(\max_{1 \leq i \leq n} \delta_i \right)^\nu, \end{aligned}$$

where $C > 0$ is the constant given by Proposition 2.1.

$$\begin{aligned} &\left| D\tilde{f}_1^n(x) - DM_1^n(x) \right| = \\ &= \left| \sum_{i=1}^n \left[DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot D\tilde{f}_i(\tilde{f}_1^{i-1}(x)) - \right. \right. \\ &\quad \left. \left. - DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x)) \right] D\tilde{f}_1^{i-1}(x) \right| \\ &\leq e^{C_4} \cdot \sum_{i=1}^n \left| DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot D\tilde{f}_i(\tilde{f}_1^{i-1}(x)) - \right. \\ &\quad \left. - DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x)) \right|. \end{aligned}$$

Now, add and subtract the term $DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x))$ in the above expression to obtain

$$\begin{aligned}
& \left| D\tilde{f}_1^n(x) - DM_1^n(x) \right| \leq \\
& \leq e^{C_4} \cdot e^{C_6} \cdot \sum_{i=1}^n \left| D\tilde{f}_i(\tilde{f}_1^{i-1}(x)) - DM_i(\tilde{f}_1^{i-1}(x)) \right| \\
& + e^{C_4} \cdot e^{C_6} \cdot \sum_{i=1}^n \left| DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) - DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \right| \\
& \leq e^{C_4} \cdot e^{C_6} \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} + e^{C_4} \cdot e^{C_6} \cdot C_7 \cdot C \cdot \sum_{i=1}^n \delta_i^{1+\nu} \\
& \leq e^{C_4} \cdot e^{C_6} \cdot (1 + C_7) \cdot C \cdot C_2 \cdot (\max_{1 \leq i \leq n} \delta_i)^\nu.
\end{aligned}$$

Now note that

$$D^2\tilde{f}_1^n(x) - D^2M_1^n(x) = \text{(I)} + \text{(II)} + \text{(III)},$$

where

$$\begin{aligned}
\text{(I)} &= \sum_{i=1}^n D^2M_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot \left(D\tilde{f}_i(\tilde{f}_1^{i-1}(x)) \right)^2 \cdot \left(D\tilde{f}_1^{i-1}(x) \right)^2 \\
&\quad - \sum_{i=1}^n D^2M_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot \left(DM_i(\tilde{f}_1^{i-1}(x)) \right)^2 \cdot \left(D\tilde{f}_1^{i-1}(x) \right)^2,
\end{aligned}$$

$$\begin{aligned}
\text{(II)} &= \sum_{i=1}^n DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot D^2\tilde{f}_i(\tilde{f}_1^{i-1}(x)) \cdot \left(D\tilde{f}_1^{i-1}(x) \right)^2 \\
&\quad - \sum_{i=1}^n DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot D^2M_i(\tilde{f}_1^{i-1}(x)) \cdot \left(D\tilde{f}_1^{i-1}(x) \right)^2
\end{aligned}$$

and

$$\begin{aligned}
\text{(III)} &= \sum_{i=1}^n DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot D\tilde{f}_i(\tilde{f}_1^{i-1}(x)) \cdot D^2\tilde{f}_1^{i-1}(x) \\
&\quad - \sum_{i=1}^n DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x)) \cdot D^2\tilde{f}_1^{i-1}(x).
\end{aligned}$$

In (I) we first add subtract the term

$$\sum_{i=1}^n D^2M_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot \left(DM_i(\tilde{f}_1^{i-1}(x)) \right)^2 \cdot \left(D\tilde{f}_1^{i-1}(x) \right)^2$$

and then we use Lemma 2.4 and Lemma 2.6 with estimates for the first derivative of $\tilde{f}_1^n$ and M_1^n , to obtain

$$(13) \quad |(I)| \leq 2 \cdot \max\{C_9, C_{10}\} \cdot \left(\max_{1 \leq i \leq n} \delta_i\right)^\nu,$$

where $C_9 = C_2 \cdot C_7 \cdot C \cdot e^{2C_4}(e^{C_4} + e^{C_6})$ and $C_{10} = C \cdot C_2 \cdot C_8 \cdot e^{2C_4} \cdot e^{C_6}$.

In (II), we first add and subtract the term

$$\sum_{i=1}^n DM_{i+1}^n(M_i \circ \tilde{f}_1^{i-1}(x)) \cdot D^2 \tilde{f}_i(\tilde{f}_1^{i-1}(x)) \cdot \left(D \tilde{f}_1^{i-1}(x)\right)^2$$

and then we use Lemma (2.4) and Lemma (2.6) with estimates for the first derivative of $\tilde{f}_1^n$ and M_1^n , to obtain

$$(14) \quad |(II)| \leq 2 \cdot \max\{C_{11}, C_{12}\} \cdot \left(\max_{1 \leq i \leq n} \delta_i\right)^\nu,$$

where $C_{11} = C \cdot C_2 \cdot C_5 \cdot C_7 \cdot e^{2C_4}$ and $C_{12} = C \cdot C_2 \cdot e^{2C_4+C_6}$.

Finally we add and subtract the expression

$$\sum_{i=1}^n DM_{i+1}^n(\tilde{f}_i \circ \tilde{f}_1^{i-1}(x)) \cdot DM_i(\tilde{f}_1^{i-1}(x)) \cdot D^2 \tilde{f}_1^{i-1}(x)$$

in (III) and use again Lemma 2.4 and Lemma 2.4, obtaining

$$(15) \quad |(III)| \leq 2 \cdot \max\{C_{13}, C_{14}\} \cdot \left(\max_{1 \leq i \leq n} \delta_i\right)^\nu,$$

where $C_{13} = C \cdot C_2 \cdot C_5 \cdot e^{C_6}$ and $C_{14} = C \cdot C_2 \cdot C_5 \cdot C_7 \cdot e^{C_6}$.

Taking $C_3 = 6 \cdot \max\{C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}\}$ we get the result. $\square$

3. RENORMALIZATION FOR GENUS ONE G.I.E.M.

Let $f \in \mathcal{B}_k^{2+\nu}$ be a g.i.e.m. with d intervals. If f has only one discontinuity then if we identify the endpoints of its domain I we obtain a piecewise smooth homeomorphism of the circle with irrational rotation. So we can apply the Denjoy Theorem for these maps, proving the non existence of wandering intervals. That is a main difference between genus on g.i.e.m. and those with higher genus (piecewise even affine g.i.e.m. with higher genus can have wandering intervals). In this section we will study this relation between genus one g.i.e.m. and piecewise homeomorphisms of the circle in more details. The combinatorial analysis next somehow extends the results in [NR97].

For the sake of simplify the notation, assume that f has only one discontinuity. Note that f can be seen as a g.i.e.m. in $f \in \mathcal{B}_k^{0+1}$ with

two intervals. Indeed, if $I_\alpha = (c_\alpha, d_\alpha)$, where d_α the the unique point of discontinuity of f , define

$$J_A := \overline{\bigcup_{\pi_0(\beta) \leq \pi_0(\alpha)} I_\beta},$$

$$J_B := \overline{\bigcup_{\pi_0(\beta) > \pi_0(\alpha)} I_\beta}.$$

Then $(f, \{A, B\}, \{J_A, J_B\})$ is a g.i.e.m. with two intervals. We can either renormalize as a g.i.e.m. with d intervals, denoted by

$$R_d(f), R_d^2(f), R_d^3(f), \dots$$

or as a g.i.e.m. with two intervals, denoted

$$R_2(f), R_2^2(f), R_2^3(f), \dots$$

We call R_d^i the i -th d -renormalization of f and R_2^i the i -th 2-renormalization of f . If we see f as a homeomorphism of the circle then we can do the usual renormalization of the circle. These sequence of renormalizations, denoted $R_{\text{rot}}^i(f)$ turns out to be just a acceleration of the Rauzy-Veech induction consisting in the subsequence of 2-renormalizations $R_2^{n_i}(f)$ defined in the following way: $n_0 = 0$ and n_{i+1} is the first $n > n_i$ whose type is distinct from the type of the n_i -th 2-renormalization.

The relation between the d and 2-renormalizations is given by the following proposition

Proposition 3.1. *Let f be a genus one g.i.e.m with d intervals in $\mathcal{B}_k$ with only one discontinuity, where $\pi_1(\alpha_0) = 1$ and $\pi_0(\alpha_1) = 1$. One of the two cases occurs*

(A) *We have*

$$\overline{\bigcup_{\pi_1(\beta) \geq \pi_1(\alpha_1)} f(I_\beta)} \subset \overline{\bigcup_{\pi_0(\beta) \geq \pi_0(\alpha_0)} I_\beta}.$$

Then f is 2-renormalizable of type 0 and $R_2(f) = R_d^n(f)$, where n is the first d -renormalization where the letter α_ wins from letter α_0 . Here α_* is such that $f(c_{\alpha_0}) \in I_{\alpha_*}$.*

(B) *We have*

$$\overline{\bigcup_{\pi_0(\beta) \geq \pi_0(\alpha_0)} I_\beta} \subset \overline{\bigcup_{\pi_1(\beta) \geq \pi_1(\alpha_1)} f(I_\beta)}.$$

Then f is 2-renormalizable of type 1 and $R_2(f) = R_d^n(f)$, where n is the first d -renormalization where the letter α_1 wins from letter α_ . Here α_* is such that $c_{\alpha_*} \in f(I_{\alpha_1})$.*

Proof. We are going to prove the Claim A. The proof of the Claim B is similar. It is easy to see that when the letter α_* wins from the letter α_0 for the first time, it wins with type 0. Using the notation defined in Eq. (1) the Rauzy-Veech algorithm is given by

$$(16) \quad (p(1), p(2), \dots, p(s), \dots, p(d)) \xrightarrow{0} (p(2), \dots, p(1), p(s), \dots, p(d)),$$

where s is such that $p(s) = 1$, and

$$(17) \quad (p(1), \dots, p(r), \dots, p(d)) \xrightarrow{1} (p(1), \dots, p(r), p(d), \dots),$$

where r is such that $p(r) = d$.

As by assumption f has only one discontinuity we have that $p = (k \dots d \ 1 \dots k - 1)$, where $\pi_1(\alpha_1) = k$.

We assert that iterating the algorithm N times, with $N = s + r \leq n - 1$, where n is such that the letter α_* wins from the letter α_0 for the first time, we obtain that

$$(18) \quad p^N = (k + s, \dots, d, \ k - r, \dots, k + s - 1, \ 1, \dots, k - r - 1),$$

where $0 \leq s \leq d - k$ and $0 \leq r \leq k - 1$ are such that

$$s = \#\{\varepsilon_m = 0 : 0 \leq m \leq N\} \quad \text{and} \quad r = \#\{\varepsilon_m = 1 : 0 \leq m \leq N\}.$$

For $N = 1$ the assertion is true because $s = 0$ and $r = 1$ or $s = 1$ and $r = 0$. Assume that the formula (18) holds for $N - 1$. Then by formulas (16) and (17) the assertion holds for N .

We know that $\varepsilon_n = 0$ and that $p^{n-1}(1) = d$. So

$$p = p^0 = (k \dots d \ 1 \dots k - 1) \xrightarrow{\varepsilon_0} \dots \xrightarrow{\varepsilon_{n-1}} p^{n-1} = (d \ j \dots d - 1 \ 1 \dots j - 1).$$

Then $p^n = (j \dots d - 1 \ d \ 1 \dots j - 1)$ which completes the proof. $\square$

Corollary 3.2. *Let f be a genus one g.i.e.m with d intervals in $\mathcal{B}_k$ with only one discontinuity. Then there exists a sequence $m_i < m_{i+1}$ such that $m_{i+1} - m_i < d$ and*

$$R_2^i(f) = R_d^{m_i}(f).$$

The next result gives us a relationship between $R_{\text{rot}}(f)$, $R_2(f)$ and $R(f)$.

Proposition 3.3. *Let f be a g.i.e.m. such that $\gamma(f)$ is k -bounded. Then for all $i \geq 0$*

$$R_{\text{rot}}^i(f) = R_2^{k_i}(f) = R_d^{n_{k_i}}(f).$$

Proof. The first equality follows by definition of R_{rot} and R_2 . The second equality follows da Proposition 3.1. $\square$

3.1. Bounded Geometry for maps in $\mathcal{B}_k^{2+\nu}$. A classical result on the circle homeomorphisms of class P (absolutely continuous homeomorphisms on the circle with bounded variation derivative) is the following lemma, whose demonstration can be found in [Her79].

Lemma 3.4. *Let f be a g.i.e.m. with genus. Let n_0 be the first n such that $R_d^{n_0} f$ has only one discontinuity and define n_i such that $R_d^{n_i} f = R_{rot}^i(R_d^{n_0} f)$. Then for all $i \geq 0$ and $\alpha \in \mathcal{A}$,*

$$\exp(-V) \leq DR_d^{n_i} f(x) \leq \exp(V) \text{ for all } x \in I_\alpha^{n_i},$$

where $V = \text{Var}(\log Df)$.

Lemma 3.5. *Let $f \in \mathcal{B}_k^{2+\mu}$. There is $C_{16} > 0$ such that*

$$\exp(-C_{16}V) \leq Df^n(x) \leq \exp(C_{16}V) \text{ for all } x \in I^n, n \in \mathbb{N}.$$

Proof. Because f has k -bounded combinatorics, there exists C with the following property: Let $i \geq 0$ be such that $n_{k_i} \leq n < n_{k_{i+1}}$. Then for every $\alpha \in \mathcal{A}$ there exists $a \leq C$ such that $(R_d^n f)(x) = (R_d^{n_{k_i}} f)^a(x)$, for $x \in I_\alpha^n$. Now the lemma follows from Lemma 3.4. $\square$

Lemma 3.6 (Non-Colapsing Domains). *Let $f \in \mathcal{B}_k^{2+\mu}$. There is $C_{17} > 1$ such that*

$$\frac{1}{C_{17}} \leq \frac{|I_\alpha^n|}{|I_\beta^n|} \leq C_{17}, \text{ for all } \alpha, \beta \in \mathcal{A} \text{ and } n \geq 0.$$

Proof. Note that by Lemma 3.5 we have that for all $\alpha \in \mathcal{A}$

$$(19) \quad \exp(-C_{16}V) \leq \frac{|R^n(f)(I_\alpha^n)|}{|I_\alpha^n|} \leq \exp(C_{16}V).$$

We claim that if the letter $\alpha_\star$ is the winner in the n -level then

$$|I_{\alpha_\star}^n|, |R^n(f)(I_{\alpha_\star}^n)| \leq \frac{k+1 - \exp(-(2^k+1)C_{16}V)}{k+1} |I^n|.$$

Indeed, otherwise by (19)

$$|I_\alpha^n|, |R^n(f)(I_\alpha^n)| < \frac{\exp(-2^k C_{16}V) |I^n|}{k+1} < \frac{\min\{|I_{\alpha_\star}^n|, |R^n(f)(I_{\alpha_\star}^n)|\}}{\exp(2^k C_{16}V)k}.$$

for every $\alpha \neq \alpha_\star$. As a consequence the letter $\alpha_\star$ will be the winner for at least k consecutive times, which contradicts $f \in \mathcal{B}_k$. So there exists $\delta \in (0, 1)$ such that

$$(20) \quad \frac{|I^{n+1}|}{|I^n|} \geq 1 - \delta$$

for every n . Note that by (19) we have

$$|I_\alpha^{n+1}| \leq \exp(C_{16}V)|I_\alpha^n|,$$

for every n and α . Moreover if $\alpha_\star$ is the winner and $\beta_\star$ is the loser at the n th level we have

$$|I_{\beta_\star}^{n+1}| \leq \exp(2C_{16}V)|I_{\alpha_\star}^n|.$$

So fix $\alpha, \beta \in \mathcal{A}$. Since β loses transitively from α after at most k renormalizations, we obtain

$$(21) \quad |I_\beta^{n+k}| \leq \exp(2kC_{16}V)|I_\alpha^n|,$$

for every n, α and β . We claim that

$$|I_\alpha^n| \geq \frac{(1-\delta)^k \exp(-2kC_{16}V)}{d} |I^n|.$$

Indeed, otherwise by (21)

$$|I^{n+k}| = \sum_{\beta} |I_\beta^n| \leq (1-\delta)^k |I^n|,$$

which contradicts (20). $\square$

Lemma 3.7 (Non-Colapsing Images). *Let $f \in \mathcal{B}_k^{2+\mu}$. There is a constant $C_{18} > 1$ such that*

$$\frac{1}{C_{18}} \leq \frac{|R^n(f)(I_\alpha^n)|}{|R^n(f)(I_\beta^n)|} \leq C_{18}, \quad \text{for all } \alpha, \beta \in \mathcal{A} \text{ and } n \geq 0.$$

Proof. Follows directly from Lemmas (3.5) and (3.6). $\square$

Proposition 3.8. *Let $f \in \mathcal{B}_k^{2+\mu}$ and let $\alpha, \beta \in \mathcal{A}$ be the winner and loser letters of $R^n(f)$, respectively. Then there is $0 < \lambda_1 < \lambda_2 < 1$, such that*

$$(22) \quad \lambda_1 < \frac{|(R^n f)^{1-\varepsilon}(I_\beta^n)|}{|(R^n f)^\varepsilon(I_\alpha^n)|} < \lambda_2,$$

where $\varepsilon \in \{0, 1\}$ is the type of $R^n(f)$.

Proof. If the quotient in (22) is too close to 0 then $(R^n f)^{1-\varepsilon}(I_\beta^n)$ is very small compared to I^n , which contradicts either Lemma 3.6 or Lemma 3.7. If the quotient in (22) is too close to 1 then $|(R^{n+1} f)^\varepsilon(I_\alpha^{n+1})| = |(R^n f)^\varepsilon(I_\alpha^n)| - |(R^n f)^{1-\varepsilon}(I_\beta^n)|$ is very small compared to I^{n+1} , what again contradicts either Lemma 3.6 or Lemma 3.7. $\square$

By definition of renormalization operator we know that

$$[0, 1) = \bigvee_{\alpha \in \mathcal{A}} \bigvee_{i=0}^{q_n^\alpha - 1} f^i(I_\alpha^n),$$

where $\bigvee$ means disjoint union. Thus the elements $f^i(I_\alpha^n)$ for all $\alpha \in \mathcal{A}$ and for all $0 \leq i \leq q_\alpha^n - 1$ form a partition which we denote by $\mathcal{P}^n$. The norm of $\mathcal{P}^n$ is defined by

$$|\mathcal{P}^n| = \max_{\substack{\alpha \in \mathcal{A} \\ 0 \leq i \leq q_\alpha^n - 1}} \{|f^i(I_\alpha^n)|\}.$$

The next result says that $|\mathcal{P}^n|$ tends to zero exponentially fast.

Proposition 3.9. *Let $f \in \mathcal{B}_k^{2+\nu}$. Then for n sufficiently large there is $\lambda = \lambda(\lambda_1, \lambda_2)$ with $0 < \lambda < 1$ such that*

$$|\mathcal{P}^{n+k}| \leq \lambda \cdot |\mathcal{P}^n|.$$

Proof. Let $f^{i_{n+k}}(I_{\alpha_n}^{n+k}) \in \mathcal{P}^{n+k}$. There are $\alpha_j \in \mathcal{A}$ and $0 \leq i_j \leq q_j^{\alpha_j} - 1$ for $n \leq j \leq n+k$ such that

$$f^{i_n}(I_{\alpha_n}^n) \supset \dots \supset f^{i_j}(I_{\alpha_j}^j) \supset f^{i_{j+1}}(I_{\alpha_{j+1}}^{j+1}) \dots \supset f^{i_{n+k}}(I_{\alpha_{n+k}}^{n+k}).$$

We claim that there exists j_0 such that α_{j_0} is the winner in the j_0 th level. Indeed if $\alpha = \alpha_n = \dots = \alpha_{n+k}$, let j_0 be a level between levels n and $n+k$ such that α wins. Such level exists because $f \in \mathcal{B}_k$. Otherwise there exists j_0 such that $\alpha_{j_0+1} \neq \alpha_{j_0}$. This is only possible if α_{j_0} is the winner and α_{j_0+1} the loser in the j_0 -th level. By Proposition 3.8 and Lemma 3.5 we have

$$\frac{|f^{i_{n+k}}(I_{\alpha_{n+k}}^{n+k})|}{|f^{i_n}(I_{\alpha_n}^n)|} \leq \frac{|f^{i_{j_0+1}}(I_{\alpha_{j_0+1}}^{j_0+1})|}{|f^{i_{j_0}}(I_{\alpha_{j_0}}^{j_0})|} \leq \lambda < 1.$$

for some λ that depends only on $V = \text{Var}(\log Df)$, λ_1 and λ_2 . $\square$

Proof of Theorem 1. Note that

$$Z_{I_\alpha^n}(R^n(f)) = Z_{f^{q_\alpha^n-1}(I_\alpha^n)}(f) \circ \dots \circ Z_{f(I_\alpha^n)}(f) \circ Z_{I_\alpha^n}(f).$$

The intervals $f^i(I_\alpha^n)$, $i = 0, \dots, q_\alpha^n - 1$ belong to the partition $\mathcal{P}^n$. In particular

$$\sum_{i=0}^{q_\alpha^n-1} |f^i(I_\alpha^n)| \leq 1$$

and by Proposition 3.9

$$\sup_{0 \leq i < q_\alpha^n} |f^i(I_\alpha^n)| \leq |\mathcal{P}^n| \leq \lambda^{n/k-1}.$$

So we can apply Proposition 2.3 to obtain that

$$|Z_{I_\alpha^n}(R^n(f)) - M_1^n|_{C^2} \leq C_3 \cdot \lambda^{\mu(n/k-1)}.$$

Recall that $M_a \circ M_b = M_{a+b}$. So

$$M_1^n = M_N,$$

where (see (8) and (3))

$$N = \sum_{i=0}^{q_\alpha^n-1} N_{Z_{f^i(I_\alpha^n)}(f)} = \sum_{i=0}^{q_\alpha^n-1} \int_{f^i(I_\alpha^n)} \frac{D^2 f(x)}{Df(x)} dx = \int_{I_\alpha^n} \frac{D^2 R^n(f)(x)}{DR^n(f)(x)} dx \quad \forall \alpha \in \mathcal{A}.$$

□

4. SYMBOLIC REPRESENTATION

To prove Theorem 2 and 3 we need a finer understanding of the geometry of the partitions generated by the sequence of renormalizations. To this end we will introduce a certain symbolic representation for the dynamics, that is somehow a generalization of the symbolic representation introduced by Sinai and Khanin [SK89]. Consider the set of *letters*

$$\mathcal{L} = \{(\alpha, \varepsilon, n) \text{ s.t. } \alpha \in \mathcal{A}, \varepsilon \in \{0, 1\}, n \in \mathbb{N}\}.$$

Define $\pi_3(\alpha, \varepsilon, n) = n$, $\pi_2(\alpha, \varepsilon, n) = \varepsilon$. We will use the notation a_i for $a_i \in \mathcal{L}$ such that $\pi_n(a_i) = i$.

In this section we construct a symbolic representation for the dynamics of a g.i.e.m. $f \in \mathcal{B}_k$. For each n we consider the partition of $[0, 1]$ given by

$$\tilde{\mathcal{P}}^n = \{f^i(I_\alpha^n) \text{ s.t. } \alpha \in \mathcal{A} \text{ and } 1 \leq i \leq q_\alpha^n\}.$$

Let

$$\Lambda = [0, 1] \setminus \cup_n \cup_{J \in \tilde{\mathcal{P}}^n} \partial J.$$

We will define a function

$$s: \Lambda \rightarrow \mathcal{L}^\mathbb{N},$$

in the following way. We have $s(x) = (a_i)_{i \in \mathbb{N}}$, $a_i \in \mathcal{L}$. If $i = 0$ then $x \in f(I_\beta^0)$ for some β and we define $a_0 = (\beta, 0, 0)$. If $i > 0$, let

$$k_{i-1}(x) := \min\{k \geq 0 \text{ s.t. } f^k(x) \in I^{i-1}\}.$$

Then either $f^{k_{i-1}}(x) \in I^i$, so $f^{k_{i-1}}(x) \in f_i(I_\beta^i)$ for some β and we define $a_i = (\beta, 0, i)$, or $f^{k_{i-1}}(x) \in I^{i-1} \setminus I^i$, so $f_{i-1}(f^{k_{i-1}}(x)) \in f_i(I_\beta^i)$ for some β and we define $a_i = (\beta, 1, i)$. Note that in any case $f^{k_i(x)}(x) \in I_\beta^i$ and $\pi_2(a_i) = 0$ if and only if $k_i(x) = k_{i-1}(x)$.

4.1. Admissible cylinders and its properties. Given a finite subset $S = \{n_1, n_2, \dots, n_k\} \subset \mathbb{N}$, with $\#S = k$ and a finite sequence $a_{n_1}, \dots, a_{n_k} \in \mathcal{L}$, with $\pi_3(a_{n_i}) = n_i$ we can consider the *word*

$$\omega = a_{n_k} a_{n_{k-1}} \dots a_{n_1}.$$

For each word we can define the cylinder

$$[\omega] = [a_{n_k} a_{n_{k-1}} \dots a_{n_1}] = \overline{\{x \in \Lambda \text{ s.t. } s_{n_i}(x) = a_{n_i}, \ 1 \leq i \leq k\}}.$$

If this cylinder is not empty we will say that the word ω is *admissible*. Indeed we can give a definition of admissible words just in terms of the combinatorial data of the g.i.e.m. f .

We claim that the set whose elements are the closures of intervals in $\tilde{\mathcal{P}}^n$ is exactly the set of admissible cylinders of the form $[a_n a_{n-1} \dots a_0]$. Indeed when $n = 0$ we have $\tilde{\mathcal{P}}^0 = \{f(I_\alpha)\}_{\alpha \in \mathcal{A}}$. Then

$$\overline{f(I_\alpha)} = [(\alpha, 0, 0)]$$

Suppose by induction that we have verified that the set of all elements $\overline{f^i(I_\alpha^n)}$, $1 \leq i \leq q_n^\alpha$, is the set of admissible cylinders of the form $[a_n a_{n-1} \dots a_0]$. Recall that $\alpha^n(\varepsilon)$, $\alpha^n(1 - \varepsilon) \in \mathcal{A}$ are the winner and loser, respectively, if f_n has type ε . There are three cases:

- If $\alpha \neq \alpha^n(1 - \varepsilon)$ then $I_\alpha^{n+1} \subset I_\alpha^n$ and $q_{n+1}^\alpha = q_n^\alpha$. So

$$\overline{f^i(I_\alpha^{n+1})} = [a_{n+1} a_n \dots a_0]$$

for every $1 \leq i \leq q_{n+1}^\alpha = q_n^\alpha$, where $a_{n+1} = (\alpha, 0, n+1)$ and $\overline{f^i(I_\alpha^n)} = [a_n \dots a_0]$;

- If $\alpha = \alpha^n(1 - \varepsilon)$ and $1 \leq i \leq q_n^{\alpha^n(\varepsilon)}$ we have

$$\overline{f^{i+(1-\varepsilon)q_n^{\alpha^n(1-\varepsilon)}}(I_{\alpha^n(1-\varepsilon)}^{n+1})} = [a_{n+1} a_n \dots a_0]$$

where $a_{n+1} = (\alpha^n(1 - \varepsilon), \varepsilon, n+1)$ and $\overline{f^i(I_{\alpha^n(\varepsilon)}^n)} = [a_n \dots a_0]$;

- If $\alpha = \alpha^n(1 - \varepsilon)$ and $1 \leq j \leq q_n^{\alpha^n(1-\varepsilon)}$ we have

$$\overline{f^{j+\varepsilon q_n^{\alpha^n(\varepsilon)}}(I_{\alpha^n(1-\varepsilon)}^{n+1})} = [a_{n+1} a_n \dots a_0],$$

where $a_{n+1} = (\alpha^n(1 - \varepsilon), 1 - \varepsilon, n+1)$ and $\overline{f^j(I_{\alpha^n(1-\varepsilon)}^n)} = [a_n \dots a_0]$.

As a consequence, for any admissible word of the form $a_n \dots a_0$, with $a_n = (\alpha, \chi, n)$ we have that the first entry times $k_1, k_2, \dots, k_n$ are constant functions in $[a_n \dots a_0]$ and $f^{k_n}[a_n \dots a_0] = f_n(I_\alpha^n)$.

Denote by $\ell(a_n \dots a_0)$ the Lebesgue measure of the cylinder $[a_n \dots a_0]$. The proof of Theorem 2 will be based on the ergodic properties of the sequence of random variables $a_n = (\alpha_n, \chi_n, n)$, $\alpha_n \in \mathcal{A}$ and $\chi_n \in \{0, 1\}$ with respect to the Lebesgue measure.

If the word $a_{n-1} \dots a_0$ is admissible we can consider the conditional probabilities

$$\ell(a_n | a_{n-1} \dots a_0) = \frac{\ell(a_n \dots a_0)}{\ell(a_{n-1} \dots a_0)}.$$

Lemma 4.1. *Let*

$$\omega_1 = a'_0 \dots a'_{n-1} a_n \dots a_{n+k},$$

$$\tilde{\omega}_1 = a''_0 \dots a''_{n-1} a_n \dots a_{n+k}$$

be admissible words. Denote $\omega_2 = a'_0 \dots a'_{n-1}$ and $\tilde{\omega}_2 = a''_0 \dots a''_{n-1}$. Then

- A. *Indeed $\pi_1(a'_{n-1}) = \pi_1(a''_{n-1}) =: \beta$ and there exist $1 \leq i, j \leq q_\beta^{n-1}$ such that $[\omega_2] = \overline{f^i(I_\beta^{n-1})}$ and $[\tilde{\omega}_2] = \overline{f^j(I_\beta^{n-1})}$,*
- B. *In particular $r = j - i$ is the unique $r \in \mathbb{Z}$ such that f^r is a diffeomorphism on $[\omega_2]$ and $f^r([\omega_2]) = [\tilde{\omega}_2]$.*
- C. *The integer r is also the unique integer such that f^r is a diffeomorphism on $[\omega_1]$ and $f^r([\omega_1]) = [\tilde{\omega}_1]$.*

Proof. The uniqueness of r follows from the fact that f does not have periodic points. Indeed if f^{r_1} and f^{r_2} , $r_1 < r_2$, are diffeomorphisms on $[\omega_i]$ and $f^r([\omega_i]) = [\tilde{\omega}_i]$ for some $i \in \{1, 2\}$ then the points in $\partial[\omega_i]$ are fixed points of $f^{r_2-r_1}$, which is a contradiction. It remains to show the existence of r . We will prove this by induction on k . Suppose $k = 0$. Denote $a_n = (\alpha, \chi, n)$. Let ε be the type of f_{n-1} . By the previous discussion about the partitions $\tilde{\mathcal{P}}^n$, there are three cases.

Case i. If $\alpha \neq \alpha^{n-1}(1 - \varepsilon)$ then $\chi = 0$ and

$$[a_n a'_{n-1} \dots a'_0] = \overline{f^i(I_\alpha^n)}$$

$$[a_n a''_{n-1} \dots a''_0] = \overline{f^j(I_\alpha^n)}$$

for some $1 \leq i, j \leq q_n^\alpha = q_{n-1}^\alpha$, with $\overline{f^i(I_\alpha^{n-1})} = [a'_{n-1} \dots a'_0]$ and $\overline{f^j(I_\alpha^{n-1})} = [a''_{n-1} \dots a''_0]$. In particular $\alpha = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. So take $r = i - j$.

Case ii. If $\alpha = \alpha^{n-1}(1 - \varepsilon)$ and $\chi = \varepsilon$ then

$$\overline{f^{i+(1-\varepsilon)q_{n-1}^\alpha}(I_\alpha^n)} = [a_n a'_{n-1} \dots a'_0],$$

$$\overline{f^{j+(1-\varepsilon)q_{n-1}^\alpha}(I_\alpha^n)} = [a_n a''_{n-1} \dots a''_0],$$

for some $1 \leq i, j \leq q_{n-1}^{\alpha^{n-1}(\varepsilon)}$ with $[a'_{n-1} \dots a'_0] = \overline{f^i(I_{\alpha^{n-1}(\varepsilon)}^{n-1})}$ and $[a''_{n-1} \dots a''_0] = \overline{f^j(I_{\alpha^{n-1}(\varepsilon)}^{n-1})}$. In particular $\alpha^{n-1}(\varepsilon) = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. So take

$$r = i - j.$$

Case *iii*. If $\alpha = \alpha^{n-1}(1 - \varepsilon)$ and $\chi = 1 - \varepsilon$ then

$$\overline{f^{i+\varepsilon q_{n-1}^{\alpha^{n-1}(\varepsilon)}}(I_\alpha^n)} = [a_n a'_{n-1} \dots a'_0],$$

$$\overline{f^{j+\varepsilon q_{n-1}^{\alpha^{n-1}(\varepsilon)}}(I_\alpha^n)} = [a_n a''_{n-1} \dots a''_0],$$

for some $1 \leq i, j \leq q_{n-1}^\alpha$ with $[a'_{n-1} \dots a'_0] = \overline{f^i(I_\alpha^{n-1})}$ and $[a''_{n-1} \dots a''_0] = \overline{f^j(I_\alpha^{n-1})}$. Again we have $\alpha = \pi_1(a'_{n-1}) = \pi_1(a''_{n-1})$. Take $r = i - j$.

This completes the proof for $k = 0$. Suppose by induction we have proved the statement for $k - 1 \geq 0$. By the case $k = 0$ there exists a unique r such that $f^r[a'_0 \dots a'_{n-1} a_n \dots a_{n+k-1}] = [a''_0 \dots a''_{n-1} a_n \dots a_{n+k-1}]$ in a diffeomorphic way, and moreover r is the unique r such that $f^r[a'_0 \dots a'_{n-1} a_n \dots a_{n+k}] = [a''_0 \dots a''_{n-1} a_n \dots a_{n+k}]$. By induction assumption there exists a unique r' such that $f^r[a'_0 \dots a'_{n-1}] = [a''_0 \dots a''_{n-1}]$ and moreover r' is the unique integer such that

$$f^r[a'_0 \dots a'_{n-1} a_n \dots a_{n+k-1}] = [a''_0 \dots a''_{n-1} a_n \dots a_{n+k-1}].$$

So $r = r'$. This completes the proof. $\square$

Lemma 4.2. *There are constants $C_{19} > 0$ and $0 < \lambda_3 = \lambda_3(\lambda) < 1$ such that*

$$e^{-C_{19}\lambda_3^s} \leq \frac{\ell(a_n | a_{n-1} \dots a_{n-s} a''_{n-s-1} \dots a''_0)}{\ell(a_n | a_{n-1} \dots a_{n-s} a'_{n-s-1} \dots a'_0)} \leq e^{C_{19}\lambda_3^s},$$

provided both words are admissible.

Proof. Let

$$f^{i_3}(I_\alpha^{n-s}) \subset f^{i_2}(I_\beta^n) \subset f^{i_1}(I_\gamma^{n+1}),$$

with $1 \leq i_3 \leq q_\alpha^{n-s}$, be the intervals corresponding to words

$$(a'_0, \dots, a'_{n-s-1}, a_{n-s}, \dots, a_n), (a'_0, \dots, a'_{n-s-1}, a_{n-s}, \dots, a_{n-1}), (a'_0, \dots, a'_{n-s-1}),$$

respectively. By Lemma 4.1 there is $j \in \mathbb{Z}$ with $1 \leq i_3 + j \leq q_{n-s}^{\pi_1(a_{n-s})}$, such that

$$f^{i_3+j}(I_\alpha^{n-s}) \subset f^{i_2+j}(I_\beta^n) \subset f^{i_1+j}(I_\gamma^{n+1})$$

are the intervals corresponding to words

$$(a''_0, \dots, a''_{n-s-1}, a_{n-s}, \dots, a_n), (a''_0, \dots, a''_{n-s-1}, a_{n-s}, \dots, a_{n-1}) \text{ and } (a''_0, \dots, a''_{n-s-1}),$$

respectively. Denote

$$\rho_k := \frac{|f^{i_2}(I_\beta^{n+1})| \cdot |f^{i_1+k}(I_\gamma^n)|}{|f^{i_1}(I_\gamma^n)| \cdot |f^{i_2+k}(I_\beta^{n+1})|}, \quad 0 \leq k \leq j.$$

We have

$$\begin{aligned}
\rho_{k+1} &:= \frac{\int_{f^{i_1+k}(I_\gamma^n)} Df(s)ds}{\int_{f^{i_2+k}(I_\beta^{n+1})} Df(s)ds} \cdot \frac{|f^{i_2}(I_\beta^{n+1})|}{|f^{i_1}(I_\gamma^n)|} \\
&= \frac{Df(s_{i_1+k})}{Df(s_{i_2+k})} \frac{|f^{i_1+k}(I_\gamma^n)|}{|f^{i_2+k}(I_\beta^{n+1})|} \cdot \frac{|f^{i_2}(I_\beta^{n+1})|}{|f^{i_1}(I_\gamma^n)|} \\
&= \frac{Df(s_{i_1+k})}{Df(s_{i_2+k})} \cdot \rho_k,
\end{aligned}$$

where $s_{i_1+k} \in f^{i_1+k}(I_\gamma^n)$ and $s_{i_2+k} \in f^{i_2+k}(I_\beta^{n+1})$. Furthermore,

$$(23) \exp\{-C_1 \cdot |f^{i_1+k}(I_\gamma^n)|\} \leq \frac{Df(s_{i_1+k})}{Df(s_{i_2+k})} \leq \exp\{C_1 \cdot |f^{i_1+k}(I_\gamma^n)|\}.$$

Then by (23) we have

$$\exp\{-C_1 \sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)|\} \leq \rho_j \leq \exp\{C_1 \sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)|\}.$$

However by Proposition 3.9,

$$\sum_{t=0}^{j-1} |f^{i_1+t}(I_\gamma^n)| = \sum_{t=0}^{j-1} |f^{i_3+t}(I_\alpha^{n-s})| \cdot \frac{|f^{i_1+t}(I_\gamma^n)|}{|f^{i_3+t}(I_\alpha^{n-s})|} \leq C_{20} \cdot \lambda_3^s,$$

where $C_{20} = C_{20}(C_{17}, C_{18}, \lambda) > 0$. Taking $C_{19} = C_1 \cdot C_{20}$ we obtain the result. $\square$

Lemma 4.3. *There exists $C_{21} = C_{21}(C_{19}, \lambda) > 0$ such that for all n, m ,*

$$e^{-C_{21}} \leq \frac{\ell(a_{n+m}, \dots, a_n | a'_{n-1}, \dots, a'_0)}{\ell(a_{n+m}, \dots, a_n | a''_{n-1}, \dots, a''_0)} \leq e^{C_{21}},$$

provided both words $(a'_0, \dots, a'_{n-1}, a_n, \dots, a_{n+m})$ and $(a''_0, \dots, a''_{n-1}, a_n, \dots, a_{n+m})$ are admissible.

Proof. The proof follows easily from Lemma 4.2 and the equations

$$\ell(a_{n+m}, \dots, a_n | a'_{n-1}, \dots, a'_0) = \prod_{i=0}^m \ell(a_{n+i} | a_{n+i-1}, \dots, a'_{n-1}, \dots, a'_0),$$

$$\ell(a_{n+m}, \dots, a_n | a''_{n-1}, \dots, a''_0) = \prod_{i=0}^m \ell(a_{n+i} | a_{n+i-1}, \dots, a''_{n-1}, \dots, a''_0).$$

□

Lemma 4.4. *Let $a_k \dots a_n$, $k \leq n$, and $a'_n a'_{n+1} \dots a'_{n+m}$ be two admissible words, such that $a_n = (\alpha, \chi, n)$, $a'_n = (\alpha, \chi', n)$. Then*

$$a_k \dots a_n a'_{n+1} \dots a'_{n+m}$$

is admissible.

Proof. Since $a_k \dots a_n$ is admissible then there exists $a_0, a_1, \dots, a_{k-1}$ such that $a_0 \dots a_{k-1} a_k \dots a_n$ is admissible. If $a'_n a'_{n+1} \dots a'_{n+m}$ is admissible then there exists an admissible word with the form $a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}$. Note that the functions $k_1(x) = k'_1$, $k_2(x) = k'_2, \dots, k_{n+m}(x) = k'_{n+m}$ are constant in $[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}]$ and

$$f^{k'_n}[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \subset f_n(I_\alpha^n).$$

The functions $k_1(x) = k_1$, $k_2(x) = k_2, \dots, k_n(x) = k_n$ are constant in $[a_0 \dots a_n]$ and

$$f^{k_n}[a_0 \dots a_n] = f_n(I_\alpha^n).$$

In particular every x in the non empty set

$$J = f^{-k_n} f^{k'_n}[a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \cap \Lambda \subset [a_0 \dots a_n]$$

belongs to the cylinder $[a_0 \dots a_n a'_{n+1} \dots a'_{n+m}]$. Indeed, since $x \in [a_0 \dots a_n]$ we have $s_i(x) = a_i$, $k_i(x) = k_i$, for $0 \leq i \leq n$. Note that $f^{k_n}(x) = f^{k'_n}(y)$, for some $y \in [a'_0 \dots a'_n a'_{n+1} \dots a'_{n+m}] \cap \Lambda$. Then

$$k_i(x) = k_i(y) - k'_n + k_n = k'_i - k'_n + k_n$$

for $n \leq i \leq n+m$, since

$$f^{k'_i - k'_n + k_n}(x) = f^{k'_i}(y) \in I^i,$$

and moreover if $k_n \leq j < k'_i - k'_n + k_n$ then

$$f^j(x) = f^{j - k_n + k'_n}(y) \notin I^i,$$

since $0 \leq j - k_n + k'_n < k'_i = k_i(y)$ and if $j < k_n$ then

$$f^j(x) \notin I^i,$$

because $j < k_n \leq k_i(x)$. This implies that $s_i(x) = a'_i$ for $n < i \leq n+m$. □

Lemma 4.5. *Let $\alpha, \beta \in \mathcal{A}$. For each n there is an admissible word $a_n \dots a_{n+k}$ with $\pi_1(a_n) = \beta$, $\pi_1(a_{n+k}) = \alpha$.*

Proof. Firstly we claim that if $\alpha \in \mathcal{A}$ does not lose in the n th level then there is an admissible word $b_n b_{n+1}$ such that $\pi_1(b_n) = \pi_1(b_{n+1}) = \alpha$. Indeed in this case $f_{n+1}(I_\alpha^{n+1}) \subset f_n(I_\alpha^n)$. This implies that the word $(\alpha, 0, n)(\alpha, 0, n+1)$ is admissible.

Second, if α loses and $\beta \in \mathcal{A}$ wins in the n th level then there is an admissible word $b_n b_{n+1}$ such that $\beta = \pi_1(b_n)$ and $\alpha = \pi_1(b_{n+1})$ and an admissible word $b'_n b'_{n+1}$ such that $\alpha = \pi_1(b'_n) = \pi_1(b'_{n+1})$. Indeed if f_n has type 1 then we have that $I_\alpha^n \subset f_n(I_\beta^n)$, I_α^n is not inside I^{n+1} and enters I^{n+1} after one iteration of f_n , landing in $f_n(I_\alpha^n) = f_{n+1}(I_\alpha^{n+1})$, so the word $(\beta, 0, n)(\alpha, 1, n+1)$ is admissible. Note also that $f_n(I_\alpha^n) = f_{n+1}(I_\alpha^{n+1})$ so $(\alpha, 0, n)(\alpha, 0, n+1)$ is admissible. if f_n has type 0 then we have that $f_{n+1}(I_\alpha^{n+1}) \subset f_n(I_\beta^n)$, so the word $(\beta, 0, n)(\alpha, 0, n+1)$ is admissible. Furthermore $f_n(I_\alpha^n)$ is not inside I^{n+1} and it enters I^{n+1} after one iteration of f_n , landing in $f_{n+1}(I_\alpha^{n+1})$, so the word $(\alpha, 0, n)(\alpha, 1, n+1)$ is admissible.

In particular, using Lemma 4.4 it follows that for every $m \geq 0$, $p > 0$ and $\alpha \in \mathcal{A}$ there exists a word $\omega = a_m a_{m+1} \dots a_{m+p}$ such that $\pi_1(a_i) = \alpha$ for every $m \leq i \leq m+p$.

Now suppose that β wins from α in the $(m-1)$ -th renormalization. Then as we saw above $(\beta, 0, m-1)(\alpha, \epsilon_{m-1}, m)$ and ω are admissible. By Lemma 4.4 there exists a word $(\beta, 0, m-1)a'_m a_{m+1} \dots a_{m+p}$ such that $\pi_1(a_{m+p}) = \alpha$.

Finally, since $f \in \mathcal{B}_k$, given $\alpha, \beta \in \mathcal{A}$, there exists a sequence of letters α_i , and levels n_i , $i \leq s$, $n \leq n_i < n_{i+1} \leq n+k$ for every $i < s$, such that $\alpha_0 = \beta$, $\alpha_s = \alpha$ and α_i wins from α_{i+1} in the n_i th level. So there are admissible words $a_{n_i}^i \dots a_{n_{i+1}}^i$ such that $\pi_1(a_{n_i}^i) = \alpha_i$ and $\pi_1(a_{n_{i+1}}^i) = \alpha_{i+1}$. By Lemma 4.4 there is an admissible word $a_{n_0} \dots a_{n_s}$ such that $\pi_1(a_{n_0}) = \beta$ and $\pi_1(a_{n_s}) = \alpha$.

Since we already proved that there exist admissible words $b_n \dots b_{n_0}$ and $c_{n_s} \dots c_{n+k}$ such that $\pi_1(b_{n_0}) = \alpha$, $\pi_1(b_{n_s}) = \alpha$, $\pi_1(c_{n_s}) = \beta$, $\pi_1(c_{n+k}) = \beta$, by Lemma 4.4 again it there exists a word of type

$$b'_n \dots b_{n_0} a_{n_0+1} \dots a_{n_s-1} c_{n_s} \dots c'_{n+k}$$

with $\pi_1(b'_n) = \alpha$ and $\pi_1(c'_{n+k}) = \beta$. $\square$

The proof of the following lemma is simple:

Lemma 4.6. *There exists $C_{22} > 0$ such that for all n, m , and all admissible words $a'_0 \dots a'_{n-k}$, $a''_0 \dots a''_{n-k}$, $a_n \dots a_{n+m}$*

$$e^{-C_{22}} \leq \frac{\ell(a_{n+m} \dots a_n | a'_{n-k} \dots a'_0)}{\ell(a_{n+m} \dots a_n | a''_{n-k} \dots a''_0)} \leq e^{C_{22}}.$$

Proposition 4.7. *There are $C_{23} > 0$ and $0 < \lambda_4 < 1$ such that*

$$|\ell(a_n|a_{n-r} \dots a_0) - \ell(a_n)| \leq C_{23} \cdot \lambda_4^{\sqrt{r}},$$

where $r = \lfloor \frac{n}{2} \rfloor$.

Proof. Indeed Proposition 4.7 is a Markov ergodic theorem and it can be proved by the methods of the theory Markov chains, as in [KS87] (see also [Sin94]). Thus, we shall describe only the main steps. Fix an integer m , $m \sim \sqrt{\frac{n}{2}}$ and introduce a new probability measure on the words of the form

$$\tilde{a} = (a_n a_{n-1} \dots a_{n-m+k} \ a_{n-m} \dots a_{n-2m+k} \ a_{n-2m} \dots a_{n-3m+k} \dots a_{n-(i-1)m} \dots a_{n-im+k} \ a_{n-im} \dots a_0)$$

by the formula

$$\begin{aligned} \ell'(\tilde{a}) &= \ell(a_0 \dots a_{n-im}) \ell(a_{n-(i-1)m} \dots a_{n-im+3} | a_{n-im} \dots a_0) \\ &\quad \prod_{s=0}^{i-2} \ell(a_{n-sm} \dots a_{n-(s+1)m+k} | a_{n-(s+1)m} \dots a_{n-(s+2)m+k}). \end{aligned}$$

Here $i \sim \sqrt{\frac{n}{2}}$. It follows easily from Lemma 4.2 that

$$(24) \quad \exp(-C_{19} \cdot \lambda_3^m \cdot i) \leq \frac{\ell'(\tilde{a})}{\ell(\tilde{a})} \leq \exp(C_{19} \cdot \lambda_3^m \cdot i).$$

The Lemma 4.6 shows that the Markov transition operator corresponding to ℓ' for the transition to m steps is a contraction for the appropriate Cayley-Hilbert metric, and this contraction is *uniformly* smaller than 1 on each step. Then the usual Ergodic Theorem for Markov chains shows that the conditional probabilities $\ell'(a_n | a_{n-im} \dots a_0)$ asymptotically do not depend on $a_{n-im} \dots a_0$. Due (24) the same holds for $\ell(a_n | a_{n-im} \dots a_0)$. This gives the desired result. $\square$

Denote by $\ell(\alpha, \star, n)$ the Lebesgue measure of the union $[(\alpha, 0, n)] \cup [(\alpha, 1, n)]$. Note that

$$\ell(\alpha, \star, n) = \frac{\sum_{i=1}^{q_\alpha^n} |f^i(I_\alpha^n)|}{|I|}.$$

Proof of Theorem 2. For simplify the notation we use f_n to denote $R^n(f)$. Let $r = \lfloor \frac{n}{2} \rfloor$. We rewrite $\int_{I_\alpha^n} \frac{D^2 f_n(s)}{D f_n(s)} ds$ in the following way. By

the mean value theorem for integrals

$$\begin{aligned} \int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)} ds &= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \int_{f^i(I_\alpha^n)} \frac{D^2 f(s)}{Df(s)} ds \\ &= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(x_j^\alpha)}{Df(x_j^\alpha)} \cdot |f^i(I_\alpha^n)|, \end{aligned}$$

where $x_i^\alpha \in f^i(I_\alpha^n)$. In a similar way we can choose $y_j^\beta \in f^j(I_\beta^r)$ such that

$$\int_I \frac{D^2 f(s)}{Df(s)} ds = \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^j(I_\beta^r)|.$$

So

$$\begin{aligned} \int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)} ds &= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \left(\frac{D^2 f(x_j)}{Df(x_j)} - \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \right) \cdot |f^i(I_\alpha^n)| \\ &\quad + \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^i(I_\alpha^n)|, \end{aligned}$$

Due to the smooth properties of f the first term is at most $C_{24} \cdot \lambda_6^{\frac{n}{2}\nu}$, where $C_{24} = C_{24}(\lambda, k) > 0$ and $0 < \lambda_6 = \lambda_6(\lambda, k) < 1$. We will now analyze the second term.

$$\begin{aligned} &\sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^i(I_\alpha^n)| = \\ &= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^j(I_\beta^r)| \frac{\sum_{f^i(I_\alpha^n) \subset f^j(I_\beta^r)} |f^i(I_\alpha^n)|}{|f^j(I_\beta^r)|} \\ &= \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^j(I_\beta^r)| \cdot \left[\ell\left((\alpha, \star, n) \mid [f^j(I_\beta^r)]\right) - \ell\left((\alpha, \star, n)\right) \right] \\ &\quad + \ell\left((\alpha, \star, n)\right) \cdot \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q_r^\beta} \frac{D^2 f(y_j^\beta)}{Df(y_j^\beta)} \cdot |f^j(I_\beta^r)| \\ &= (\text{IV}) + (\text{V}). \end{aligned}$$

By Proposition 4.7 we have that (IV) = $O(\lambda_4^{\sqrt{\frac{n}{2}}})$. Now observe that (V) is a Riemann sum for the integral $\int_I \frac{D^2 f(s)}{Df(s)} ds$. By Proposition 3.9 and ν -Holder continuity of $\frac{D^2 f}{Df}$ we have

$$(V) = \ell((\alpha, \star, n)) \cdot \int_I \frac{D^2 f(s)}{Df(s)} ds + O(\lambda^{n/k}).$$

This finishes the proof. $\square$

Before proving the Theorem 3 we need the following lemma whose proof is left to the reader.

Lemma 4.8. *Let $a, b \in \mathbb{R}$. Then for every $C > 0$ there is $C_{25} > 0$ such that if $|a|, |b| \leq C$ then*

$$|M_a - M_b|_{C^2} \leq C_{25} \cdot |a - b|,$$

where M_a and M_b are defined in (2).

Proof of Theorem 3. By assumption $\int_I \frac{D^2 f(s)}{Df(s)} ds = 0$, so by Theorem 2 we have

$$\left| \int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)} ds \right| \leq C_{26} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}.$$

Therefore by Lemma 4.8

$$(25) \quad \left| M_{\int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)} ds} - \text{Id} \right|_{C^2} \leq C_{25} \cdot \left| \int_{I_\alpha^n} \frac{D^2 f_n(s)}{Df_n(s)} ds - 0 \right| \\ \leq C_{25} \cdot C_{26} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}.$$

Theorem 1 together with (25) gives us that

$$|\mathcal{Z}_{I_\alpha^n}(R^n(f)) - \text{Id}|_{C^2} \leq C_{27} \cdot \lambda_4^{\sqrt{\frac{n}{2}}}.$$

$\square$

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