

# Non-distributive algebraic structures derived from nonextensive statistical mechanics

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## Abstract.

We propose a two-parametric non-distributive algebraic structure that follows from  $(q, q')$ -logarithm and  $(q, q')$ -exponential functions. Properties of generalized  $(q, q')$ -operators are analyzed. We also generalize the proposal into a multi-parametric structure (generalization of logarithm and exponential functions and the corresponding algebraic operators). All  $n$ -parameter expressions recover  $(n - 1)$ -generalization when the corresponding  $q_n \rightarrow 1$ . Nonextensive statistical mechanics has been the source of successive generalizations of entropic forms and mathematical structures, in which this work is a consequence.

**Keywords:** Nonextensive statistical mechanics, deformed functions, deformed algebraic structures

## 1. Introduction

The  $q$ -logarithm and the  $q$ -exponential functions [1] appears in the very foundations of nonextensive statistical mechanics [2], and they are present in (virtually) every development of the theory and its applications. These functions are generalizations of the usual logarithm and exponential, given by

$$\ln_q x := \frac{x^{1-q} - 1}{1 - q}, \quad x > 0, \quad (1)$$

and

$$e_q^x := [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \quad (2)$$

where  $[p]_+ := \max\{p, 0\}$ . They are inverse of each other, and recover their usual counterparts as  $q \rightarrow 1$ . The  $q$ -logarithm function allows the nonextensive entropy be neatly written as

$$S_q = k \sum_{i=1}^W p_i \ln_q(1/p_i), \quad (3)$$

and also the  $q$ -canonical distribution is given by a  $q$ -exponential function, making a close formal parallelism with Boltzmann-Gibbs statistical mechanics.

The  $q$ -logarithm and  $q$ -exponential functions have led to the development of a  $q$ -algebra [3, 4], by the definition of a  $q$ -sum

$$x \oplus_q y := x + y + (1 - q)xy \quad (4)$$

and a  $q$ -product

$$x \otimes_q y := [x^{1-q} + y^{1-q} - 1]_+^{\frac{1}{1-q}}, \quad x > 0, y > 0. \quad (5)$$

These operators allow to write properties of  $\ln_q x$  and  $e_q^x$  in a simple and compact form, *e.g.*,

$$\ln_q(xy) = \ln_q x \oplus_q \ln_q y, \quad (6)$$

$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y, \quad (7)$$

and correspondingly,

$$e_q^x e_q^y = e_q^{x \oplus_q y}, \quad (8)$$

$$e_q^{x+y} = e_q^x \otimes_q e_q^y. \quad (9)$$

It is possible to define the inverse operators  $q$ -difference and  $q$ -ratio (and  $q$ -power,  $q$ -root), yielding a non-distributive  $q$ -algebraic structure.

There have been proposals of further generalizations of either  $S_q$  or  $q$ -logarithm and  $q$ -exponential functions (or both), by means of two parameters. One instance was proposed by Papa [5], who developed an infinite set of entropies  ${}_n S_q$  ( $n = 1, 2, \dots$  is called order of the entropy).  $S_q$  entropy is recovered with  $n = 1$ , and Boltzmann-Gibbs entropy is recovered by either  $q \rightarrow 1$  or  $n \rightarrow \infty$ .

A two-parameter  $(q, q')$ -entropy  $S_{q, q'}$  was proposed by Roditi and one of us [6]; it is based on Chakrabarti and Jagannathan two-parameter derivative [7], following Abe's rule [8] for generating entropies. This  $S_{q, q'}$  entropy is symmetric by exchange of indexes  $q \leftrightarrow q'$  and recovers nonextensive entropy  $S_q$  if either  $q$  or  $q'$  is set to unity. From  $S_{q, q'}$ , the authors define a two-parameter  $(q, q')$ -logarithm — its inverse, the  $(q, q')$ -exponential, has an implicit form.  $S_{q, q'}$  is, in fact, a simple variation of Mittal two-parameter entropy [9], proposed years before the nonextensive seminal paper [2], as remarked by [10].

Kaniadakis and co-authors [10] propose a different two-parameter entropy, not immediately connected to nonextensive  $S_q$ , and coherently introduce a two-parameter  $(\kappa, r)$ -deformed logarithm and exponential functions. Their work is a consequence of previous papers, starting with [11], in which it is introduced the  $\kappa$ -exponential, that

satisfies  $\exp_{\kappa}^{-1}(x) = \exp_{\kappa}(-x)$  (differently from the  $q$ -exponential given by Eq. (2), which obeys  $e_q^{-1}(x) = e_q^{(-x/(1+(1-q)x))}$ ).

Along a different path, Tsallis, Bemsiki and Mendes [12] introduced a two-parameter generalization of the  $q$ -exponential, Eq. (2): the successive generalization of the exponential function  $(e^x, e_q^x, e_{q,1}^x, e_{q,q'}^x)$  is obtained by the generalization of the differential equation it obeys. Their  $(q,q')$ -exponential is presented as a function of hypergeometric functions (it cannot be expressed explicitly), and recover the one-parameter  $e_q^x$  as  $q = q'$ . This  $(q,q')$ -exponential presents two power law regimes, with a crossover between them (as a particular case, with  $q' = 1$ , it can exhibit a crossover from a power-law regime to an exponential tail), and this interesting feature has been observed in a variety of phenomena [12, 13, 14, 15, 16].

Recently a still different proposal have appeared in literature: Schwämmle and Tsallis [17] introduce a two-parameter generalization of the logarithm and exponential functions. The  $(q,q')$ -logarithm is properly defined in order to satisfy

$$\ln_{q,q'}(x \otimes_q y) = \ln_{q,q'} x \oplus_{q'} \ln_{q,q'} y, \quad (10)$$

which is a generalization of two basic properties of the one-parameter  $q$ -logarithm, Eq. (6)–(7), and, correspondingly, the  $(q,q')$ -exponential satisfies

$$e_{q,q'}^{x \oplus_{q'} y} = e_{q,q'}^x \otimes_q e_{q,q'}^y, \quad (11)$$

that generalizes Eq. (8)–(9).

In our present work, we use Schwämmle and Tsallis  $(q,q')$ -deformed functions and define two-parameter algebraic operators, following the lines of [10] and generalizing [3, 4].

It is usual that different forms of deformed functions receive the same notation and are called by similar names,  $q$ -exponential,  $(q,q')$ -exponential, and so on; but this shall not confound the reader, and (s)he must be able to distinguish between them by the context.

The paper is organized as follows: Sec. 2 introduces a new two-parameter algebra and presents some of its properties; Sec. 3 proposes a different two-parameter deformation of the product; multi-parametric generalization of functions and algebras are addressed in Sec. 4. Finally, in Sec. 5 we draw our conclusions and present some perspectives. The relative simplicity of the two-parameter equations, when expressed by means of the  $q$ -deformed versions, as they appear in the main body of the paper, contrast with the explicit forms (presented in the Appendix) that are not so simple and clean.

## 2. A Two-Parameter Algebra

Let us rewrite the basic properties  $e^{x+y} = e^x e^y$  and  $\ln(xy) = \ln x + \ln y$  as

$$\begin{aligned} x + y &\equiv \ln(e^x e^y), \\ xy &\equiv e^{\ln x + \ln y}. \end{aligned} \quad (12)$$

We may take these expressions as a basis for a redefinition of the  $q$ -algebraic operators: the  $q$ -sum (Eq. (4)) and the  $q$ -product (Eq. (5)) can be found by the following recipe: replace the logarithm and exponential of Eq. (12) by their  $q$ -generalizations:

$$\begin{aligned} x \oplus_q y &:= \ln_q(e_q^x e_q^y), \\ x \otimes_q y &:= e_q^{\ln_q x + \ln_q y}. \end{aligned} \quad (13)$$

In Ref. [18], where this recipe was proposed, it was shown that other pairs of function–inverse function can be used (and not only the logarithm–exponential pair), and thus yielding different deformed algebraic operators. We shall apply this recipe in the following.

The two-parameter generalization of the  $q$ -logarithm and  $q$ -exponential functions introduced in [17] can be written as (see Eq. (A.1) and (A.2) in the Appendix for the explicit forms)

$$\ln_{q,q'} x = \ln_{q'} e^{\ln_q x}, \quad (14)$$

$$e_{q,q'}^x = e_q^{\ln e_{q'}^x}. \quad (15)$$

Though these  $(q,q')$ -functions are *not* symmetric by interchange of parameters  $q \leftrightarrow q'$ , the mono-parametric  $q$ -functions are recovered by either  $q \rightarrow 1$  or  $q' \rightarrow 1$ . They are inverse of each other, as may be promptly verified by  $\ln_{q,q'} e_{q,q'}^x = e_{q,q'}^{\ln_{q,q'} x} = x$ . Ref. [17] brings various properties of these functions.

Here we propose a natural extension of Eq.s (13) for a two-parameter version:

$$x \oplus_{q,q'} y := \ln_{q,q'}(e_{q,q'}^x e_{q,q'}^y) \quad (16)$$

and

$$x \otimes_{q,q'} y := e_{q,q'}^{\ln_{q,q'} x + \ln_{q,q'} y}. \quad (17)$$

It is possible to rewrite the  $(q,q')$ -sum and the  $(q,q')$ -product expressing them by means of the (mono-parametric)  $q$ -functions:

$$x \oplus_{q,q'} y = \ln_{q'} e^{\ln e_{q'}^x \oplus_q \ln e_{q'}^y}, \quad (18)$$

with  $1 + (1 - q')x > 0$ ,  $1 + (1 - q')y > 0$ , and

$$x \otimes_{q,q'} y = e_q^{\ln(e^{\ln_q x} \otimes_{q'} e^{\ln_q y})}, \quad (19)$$

with  $x > 0$ ,  $y > 0$ ,  $[(e^{\ln_q x})^{1-q'} + (e^{\ln_q y})^{1-q'}] > 1$ . Explicit expressions are, respectively, Eq. (A.3) and Eq. (A.4).

Let us address some properties of the  $(q,q')$ -sum: it is commutative,

$$x \oplus_{q,q'} y = y \oplus_{q,q'} x, \quad (20)$$

associative,

$$x \oplus_{q,q'} (y \oplus_{q,q'} z) = (x \oplus_{q,q'} y) \oplus_{q,q'} z, \quad (21)$$

and presents neutral element:

$$x \oplus_{q,q'} 0 = x, \forall q, q' \in \mathbb{R}. \quad (22)$$

The opposite element is given by:

$$x \oplus_{q,q'} (\ominus_{q,q'} x) = 0, \quad (23)$$

which implies (see also (A.5))

$$\ominus_{q,q'} x = \ln_{q'} e^{\ominus_q(\ln e_{q'}^x)}, \quad (24)$$

with  $1 + (1 - q')x > 0$ ,  $x \neq \ln_{q'} e^{\frac{-1}{1-q}}$ .

Using Eq. (24) we can define the  $(q, q')$ -difference as:

$$x \ominus_{q,q'} y := x \oplus_{q,q'} (\ominus_{q,q'} y) = \ln_{q'} e^{\ln e_{q'} x \ominus_q \ln e_{q'} y} \quad (25)$$

with  $1 + (1 - q')x > 0$ ,  $1 + (1 - q')y > 0$ ,  $y \neq \ln_{q'} e^{\frac{-1}{1-q}}$  (see Eq. (A.6)), which is a generalization of the  $q$ -difference operator [3, 4],

$$x \ominus_q y = \frac{x - y}{1 + (1 - q)y}, \quad y \neq \frac{-1}{1 - q}. \quad (26)$$

The  $(q, q')$ -sum is non-distributive:

$$a(x \oplus_{q,q'} y) \neq (ax) \oplus_{q,q'} (ay), \quad \forall a \neq 0, a \neq 1, \forall q, q' \in \mathbb{R}. \quad (27)$$

Now we present the corresponding properties of the  $(q, q')$ -product: commutativity,

$$x \otimes_{q,q'} y = y \otimes_{q,q'} x, \quad (28)$$

associativity,

$$x \otimes_{q,q'} (y \otimes_{q,q'} z) = (x \otimes_{q,q'} y) \otimes_{q,q'} z, \quad (29)$$

it presents neutral element,

$$x \otimes_{q,q'} 1 = x, \quad \forall q, q' \in \mathbb{R}, \quad (30)$$

and its inverse is given by

$$x \otimes_{q,q'} (1 \oslash_{q,q'} x) = 1, \quad (31)$$

which implies (see also Eq. (A.7))

$$1 \oslash_{q,q'} x = e_q^{\ln(1 \oslash_{q'} e^{\ln_q x})}, \quad (32)$$

with  $0 < x < e_q^{\frac{\ln 2}{1-q}}$  for  $q' < 1$ , and  $x > e_q^{-\frac{\ln 2}{q'-1}}$  for  $q' > 1$ .

Using Eq. (32) we can define the  $(q, q')$ -ratio as (see Eq. (A.8)):

$$x \oslash_{q,q'} y := x \otimes_{q,q'} (1 \oslash_{q,q'} y) = e_q^{\ln(e^{\ln_q x} \oslash_{q'} e^{\ln_q y})}, \quad (33)$$

with  $x > 0$ ,  $y > 0$ ,  $[(e^{\ln_q x})^{1-q'} - (e^{\ln_q y})^{1-q'} + 1] > 0$ . This equation recovers the  $q$ -ratio as  $q' \rightarrow 1$  (or  $q \rightarrow 1$ ):

$$x \oslash_q y = [x^{1-q} - y^{1-q} + 1]_+^{\frac{1}{1-q}}, \quad x > 0, y > 0. \quad (34)$$

The  $(q, q')$ -product is non-distributive:

$$a \otimes_{q,q'} (x + y) \neq (a \otimes_{q,q'} x) + (a \otimes_{q,q'} y), \quad \forall a \neq 1, \forall q, q' \in \mathbb{R}. \quad (35)$$

The  $(q, q')$ -logarithm and  $(q, q')$ -exponential satisfy the following relations, expressed by means of these new two-parameter operators (restrictions of  $(q, q')$ -operators and  $(q, q')$ -functions shall apply):

$$\begin{aligned}
 \ln_{q, q'}(xy) &= \ln_{q, q'} x \oplus_{q, q'} \ln_{q, q'} y & e_{q, q'}^{x \oplus_{q, q'} y} &= e_{q, q'}^x e_{q, q'}^y \\
 \ln_{q, q'}(x \otimes_{q, q'} y) &= \ln_{q, q'} x + \ln_{q, q'} y & e_{q, q'}^{x+y} &= e_{q, q'}^x \otimes_{q, q'} e_{q, q'}^y \\
 \ln_{q, q'}(x/y) &= \ln_{q, q'} x \ominus_{q, q'} \ln_{q, q'} y & e_{q, q'}^{x \ominus_{q, q'} y} &= e_{q, q'}^x / e_{q, q'}^y \\
 \ln_{q, q'}(x \oslash_{q, q'} y) &= \ln_{q, q'} x - \ln_{q, q'} y & e_{q, q'}^{x-y} &= e_{q, q'}^x \oslash_{q, q'} e_{q, q'}^y
 \end{aligned} \tag{36}$$

These  $(q, q')$ -operators are also *not* symmetric in relation to the interchange of parameters  $q \leftrightarrow q'$ , and they are reduced to the mono-parametric  $q$ -operators if either  $q$  or  $q'$  are set to unity, similarly to the behavior exhibited by the  $(q, q')$ -functions that have originated them.

### 3. The $(q, q')$ -Dot Product

A different deformed product, introduced in [4], is defined by repeated  $q$ -sums of  $n$  ( $n \in \mathbb{N}$ ) equal terms:

$$n \odot_q x := \underbrace{x \oplus_q x \oplus_q \cdots \oplus_q x}_{n \text{ times}} = \frac{[1 + (1 - q)x]_+^n - 1}{1 - q}. \tag{37}$$

Of course we have  $n \odot_{q \rightarrow 1} x = nx$ . It is possible to extend its definition from  $\odot_q : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$  to  $\odot_q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ :

$$a \odot_q x := \frac{[1 + (1 - q)x]_+^a - 1}{1 - q}. \tag{38}$$

This product is not commutative, however it determines a one-dimensional vector space structure over  $\mathbb{R}$  if we observe that

$$a \odot_q (x \oplus_q y) = (a \odot_q x) \oplus_q (a \odot_q y), \tag{39}$$

and

$$(a + b) \odot_q x = (a \odot_q x) \oplus_q (b \odot_q x) \tag{40}$$

( $a, b, x, y \in \mathbb{R}$ ).

The  $q$ -dot product may be expressed by means of the  $q$ -exponential and  $q$ -logarithm functions:

$$a \odot_q x = \ln_q[(e_q^x)^a]. \tag{41}$$

An interesting property of the  $q$ -dot product is

$$e_q^x = [e_q^{(a \odot_q x)}]_a^{\frac{1}{a}}. \tag{42}$$

We can proceed analogously and define the  $(q, q')$ -dot product as:

$$n \odot_{q, q'} x := \underbrace{x \oplus_{q, q'} x \oplus_{q, q'} \cdots \oplus_{q, q'} x}_{n \text{ times}} \quad (n \in \mathbb{N}), \tag{43}$$

that may be expressed as

$$n \odot_{q,q'} x = \ln_{q,q'} [(e_{q,q'}^x)^n], \quad (44)$$

and extended to  $\odot_q : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , turning into

$$a \odot_{q,q'} x = \ln_{q,q'} [(e_{q,q'}^x)^y] \quad (45)$$

(see Eq. (A.9)).

Some properties of the  $(q, q')$ -dot product  $(a, x, y \in \mathbb{R})$ :

$$a \odot_{q,q'} x = \ln_{q'} e^{a \odot_q \ln e_{q'}^x}, \quad (46)$$

$$e_{q,q'}^x = [e_{q,q'}^{(a \odot_{q,q'} x)}]^{\frac{1}{a}}, \quad (47)$$

$$a \odot_{q,q'} (x \oplus_{q,q'} y) = (a \odot_{q,q'} x) \oplus_{q,q'} (a \odot_{q,q'} y), \quad (48)$$

$$(a + b) \odot_{q,q'} x = (a \odot_{q,q'} x) \oplus_{q,q'} (b \odot_{q,q'} x). \quad (49)$$

#### 4. Multi-parametric Functions and Algebras

We can further generalize logarithms and exponentials into a multi-parametric family of functions, and correspondingly, generate multi-parametric algebraic structures. For instance, Eq. (14) can be generalized into three parameters as<sup>‡</sup>:

$$\ln_{q_1, q_2, q_3} x := \ln_{q_3} \exp(\ln_{q_1, q_2} x). \quad (50)$$

The procedure can be extended to an arbitrary number  $n$  of parameters: let us consider a set of  $n$  parameters  $\{q_1, q_2, \dots, q_n\}$ , symbolically represented by  $\{q_n\}$ , and let us define the  $n$ -parameter generalization of the logarithm function in terms of the  $(n-1)$ -parameter logarithm

$$\ln_{\{q_n\}} x := \ln_{q_n} \exp(\ln_{\{q_{n-1}\}} x), \quad n > 1. \quad (51)$$

Note that  $\ln_{q_n} x$  is the mono-parametric  $q$ -logarithm with  $q = q_n$ . The  $n$ -parameter generalization of the exponential function must be consistently defined as the inverse function of Eq. (51),

$$\ln_{\{q_n\}} \exp_{\{q_n\}}(x) = \exp_{\{q_n\}}(\ln_{\{q_n\}} x) = x, \quad (52)$$

and thus,

$$\exp_{\{q_n\}}(x) := \exp_{\{q_{n-1}\}}(\ln \exp_{q_n}(x)). \quad (53)$$

Now we can promptly apply the recipes given by Eq.s (12), (13), and (16)–(17), and define

$$x \oplus_{\{q_n\}} y := \ln_{\{q_n\}} (\exp_{\{q_n\}}(x) \exp_{\{q_n\}}(y)), \quad (54)$$

$$x \otimes_{\{q_n\}} y := \exp_{\{q_n\}} (\ln_{\{q_n\}} x + \ln_{\{q_n\}} y). \quad (55)$$

<sup>‡</sup> We adopt the notation  $\exp_q(x)$  in some of the equations of this section for aesthetical reasons.

With these operators, Eq.s (6)–(9) are generalized into:

$$\ln_{\{q_n\}}(xy) = \ln_{\{q_n\}} x \oplus_{\{q_n\}} \ln_{\{q_n\}} y, \quad (56)$$

$$\ln_{\{q_n\}}(x \otimes_{\{q_n\}} y) = \ln_{\{q_n\}} x + \ln_{\{q_n\}} y, \quad (57)$$

$$e_{\{q_n\}}^x e_{\{q_n\}}^y = e_{\{q_n\}}^{x \oplus_{\{q_n\}} y}, \quad (58)$$

$$e_{\{q_n\}}^{x+y} = e_{\{q_n\}}^x \otimes_{\{q_n\}} e_{\{q_n\}}^y. \quad (59)$$

The motivation of Ref. [17] is still preserved: the generalizations of Eq. (56)–(57) and Eq. (58)–(59) are:

$$\ln_{\{q_{m+n}\}}(x \otimes_{q_1, \dots, q_m} y) = \ln_{\{q_{m+n}\}} x \oplus_{q_{m+1}, \dots, q_{m+n}} \ln_{\{q_{m+n}\}} y, \quad (60)$$

$$\exp_{\{q_{m+n}\}}(x \oplus_{q_{m+1}, \dots, q_{m+n}} y) = \exp_{\{q_{m+n}\}}(x) \otimes_{q_1, \dots, q_m} \exp_{\{q_{m+n}\}}(y). \quad (61)$$

In the former equations,  $\{q_{m+n}\} \equiv \{q_1, \dots, q_m, q_{m+1}, \dots, q_n\}$ , and the order of the parameters cannot be changed.

It is straightforward to define the  $\{q_n\}$ -inverse operators  $\{q_n\}$ -difference,  $\{q_n\}$ -ratio, as well as the  $\{q_n\}$ -power operator; we shall not explicit these relations here for the sake of brevity.

Analogously we can define multi-parametric  $\{q_n\}$ -dot-product by applying the recipe suggested by Eq. (45):

$$a \odot_{\{q_n\}} x := \ln_{\{q_n\}} [\exp_{\{q_n\}}(x)]^a. \quad (62)$$

Properties of  $\odot_{\{q_n\}}$  are straightforwardly analogous to those of  $\odot_{q, q'}$ :

$$a \odot_{\{q_n\}} x = \ln_{q_n} e^{a \odot_{\{q_{n-1}\}} \ln e_{q_n}^x}, \quad (63)$$

$$a \odot_{\{q_{n+m}\}} x = \ln_{q_{n+1}, \dots, q_{n+m}} e^{a \odot_{q_1, \dots, q_n} \ln e_{q_{n+1}, \dots, q_{n+m}}^x}, \quad (64)$$

$$e_{\{q_{n+m}\}}^x = \left[ e_{q_1, \dots, q_n}^{a \odot_{q_1, \dots, q_n} \ln e_{q_{n+1}, \dots, q_{n+m}}^x} \right]^{\frac{1}{a}}, \quad (65)$$

$$a \odot_{\{q_n\}} (x \oplus_{\{q_n\}} y) = (a \odot_{\{q_n\}} x) \oplus_{\{q_n\}} (a \odot_{\{q_n\}} y), \quad (66)$$

$$(a + b) \odot_{\{q_n\}} x = (a \odot_{\{q_n\}} x) \oplus_{\{q_n\}} (b \odot_{\{q_n\}} x). \quad (67)$$

## 5. Conclusions and Perspectives

We have introduced a two-parameter  $(q, q')$ -algebraic structure, that naturally emerges from the recently defined  $(q, q')$ -logarithm and  $(q, q')$ -exponential [17], and we have investigated some of its properties. The developments follows the lines of Ref. [3, 4], in which it was introduced the mono-parametric  $q$ -algebra. We have also introduced the  $(q, q')$ -dot product, defined by repeated  $(q, q')$ -sums. Besides, we have extended the procedure, and defined multi-parametric logarithm and exponential functions, and their corresponding algebras.

The possible applications of these developments are not still clear, but we recall the central role played by the  $q$ -product in the generalization of the central limit theorem

and the  $q$ -Fourier transform [19, 20, 21, 22, 23]; it remains as a possibility the use of  $(q, q')$ -product for further developments of these theories. We also recall that in the developments of the generalization of the central limit theorem, it appears not only one, but a sequence of  $q_n$  parameters [19] (or  $q_{\alpha, n}$  parameters, as in [21]). The existence of a  $q$ -triplet  $(q_{sen}, q_{rel}, q_{stat})$  was conjectured in [24], and it was observed in the solar wind at the distant heliosphere [25]. The connections of the theories that use multiple entropic indexes with the developments here introduced is a possibility to be further investigated.

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## Appendix

Here we present explicit (and unfortunately cumbersome) forms of some expressions introduced in this paper.

**$(q, q')$ -logarithm** (Eq. (14)):

$$\ln_{q, q'} x = \frac{1}{1 - q'} \left\{ \exp \left[ \frac{1 - q'}{1 - q} (x^{1-q} - 1) \right] - 1 \right\}, \quad x > 0. \quad (\text{A.1})$$

**$(q, q')$ -exponential** (Eq. (15)):

$$e_{q, q'}^x = \left\{ 1 + \frac{1 - q}{1 - q'} \ln[1 + (1 - q')x] \right\}_+^{\frac{1}{1-q}}, \quad [1 + (1 - q')x] > 0. \quad (\text{A.2})$$

**$(q, q')$ -sum** (Eq. (16)):

$$x \oplus_{q, q'} y = \frac{\omega_{q'}(x)\omega_{q'}(y) \exp \left\{ \left( \frac{1-q}{1-q'} \right) \ln[\omega_{q'}(x)] \ln[\omega_{q'}(y)] \right\} - 1}{1 - q'}, \quad (\text{A.3})$$

with  $\omega_{q'}(u) \equiv 1 + (1 - q')u$ ,  $\omega_{q'}(u) > 0$ .

**$(q, q')$ -product** (Eq. (17)):

$$x \otimes_{q, q'} y = \left\{ 1 + \left( \frac{1-q}{1-q'} \right) \ln \left[ \left( e^{\frac{x^{1-q}-1}{1-q}} \right)^{1-q'} + \left( e^{\frac{y^{1-q}-1}{1-q}} \right)^{1-q'} - 1 \right] \right\}_+^{\frac{1}{1-q}}, \quad (\text{A.4})$$

with  $x > 0$ ,  $y > 0$ ,  $\left[ \left( e^{\frac{x^{1-q}-1}{1-q}} \right)^{1-q'} + \left( e^{\frac{y^{1-q}-1}{1-q}} \right)^{1-q'} - 1 \right] > 0$ .

**$(q, q')$ -opposite** (Eq. (24)):

$$\ominus_{q, q'} y = \frac{\exp \left\{ \frac{-\ln[1+(1-q')y]}{1 + \left( \frac{1-q}{1-q'} \right) \ln[1+(1-q')y]} \right\} - 1}{1 - q'}, \quad (\text{A.5})$$

with  $[1 + (1 - q')y] > 0$ ,  $y \neq \frac{e^{\frac{1-q'}{1-q}} - 1}{1 - q'}$ .

**( $q, q'$ )-difference** (Eq. (25)):

$$x \ominus_{q,q'} y = \frac{\exp \left\{ \frac{\ln \left[ \frac{1+(1-q')x}{1+(1-q')y} \right]}{1 + \left( \frac{1-q}{1-q'} \right) \ln [1+(1-q')y]} \right\} - 1}{1 - q'}, \quad (\text{A.6})$$

with  $[1 + (1 - q')x] > 0$ ,  $[1 + (1 - q')y] > 0$ ,  $y \neq \frac{e^{\frac{1-q'}{1-q}} - 1}{1-q}$ .

**( $q, q'$ )-inverse** (Eq. (32)):

$$1 \oslash_{q,q'} y = \left\{ 1 + \left( \frac{1-q}{1-q'} \right) \ln \left[ 2 - \left( e^{\frac{y^{1-q}-1}{1-q}} \right)^{1-q'} \right] \right\}_+^{\frac{1}{1-q}}, \quad (\text{A.7})$$

with  $0 < y < \left[ 1 + \frac{1-q}{1-q'} \ln 2 \right]^{\frac{1}{1-q}}$  for  $q' < 1$ , and  $y > \left[ 1 + \frac{1-q}{1-q'} \ln 2 \right]^{\frac{1}{1-q}}$  for  $q' > 1$ .

**( $q, q'$ )-ratio** (Eq. (33)):

$$x \otimes_{q,q'} y = \left\{ 1 + \left( \frac{1-q}{1-q'} \right) \ln \left[ \left( e^{\frac{x^{1-q}-1}{1-q}} \right)^{1-q'} - \left( e^{\frac{y^{1-q}-1}{1-q}} \right)^{1-q'} + 1 \right] \right\}_+^{\frac{1}{1-q}}, \quad (\text{A.8})$$

with  $x > 0$ ,  $y > 0$ ,  $\left[ \left( e^{\frac{x^{1-q}-1}{1-q}} \right)^{1-q'} - \left( e^{\frac{y^{1-q}-1}{1-q}} \right)^{1-q'} + 1 \right] > 0$ .

**( $q, q'$ )-dot-product** (Eq. (45)):

$$a \odot_{q,q'} x = \frac{\exp \left\{ \left( \frac{1-q'}{1-q} \right) \left[ \left[ 1 + \left( \frac{1-q}{1-q'} \right) \ln (1+(1-q')x) \right]_+^a - 1 \right] \right\} - 1}{1 - q'}, \quad (\text{A.9})$$

with  $[1 + (1 - q')x] > 0$ .

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