# *Infinitesimal Lyapunov functions for singular flows*

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# Infinitesimal Lyapunov functions for singular flows

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**Abstract** We present an extension of the notion of infinitesimal Lyapunov function to singular flows, and from this technique we deduce a characterization of partial/sectional hyperbolic sets. In absence of singularities, we can also characterize uniform hyperbolicity. These conditions can be expressed using the space derivative $DX$ of the vector field $X$ together with a field of infinitesimal Lyapunov functions only, and are reduced to checking that a certain symmetric operator is positive definite at the tangent space of every point of the trapping region.

**Keywords** Dominated splitting · Partial hyperbolicity · Sectional hyperbolicity · Infinitesimal Lyapunov function

**Mathematics Subject Classification (2000)** Primary 37D30; Secondary 37D25

## 1 Introduction

The hyperbolic theory of dynamical systems is now almost a classical subject in mathematics and one of the main paradigms in dynamics. Developed in the 1960s and 1970s after the work of Smale, Sinai, Ruelle, Bowen [11,12,42,43], among many others, this theory deals

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with compact invariant sets $\Lambda$ for diffeomorphisms and flows of closed finite-dimensional manifolds having a hyperbolic splitting of the tangent space. That is, $T_\Lambda M = E^s \oplus E^X \oplus E^u$ is a continuous splitting of the tangent bundle over $\Lambda$, where $E^X$ is the flow direction, the subbundles are invariant under the derivative $DX_t$ of the flow $X_t$

$$DX_t \cdot E^*_x = E^*_{X_t(x)}, \quad x \in \Lambda, \quad t \in \mathbb{R}, \quad * = s, X, u;$$

$E^s$ is uniformly contracted by $DX_t$ and $E^u$ is likewise expanded: there are $K, \lambda > 0$ so that

$$\|DX_t \mid E^s_x\| \le Ke^{-\lambda t}, \quad \|(DX_t \mid E^u_x)^{-1}\| \le Ke^{-\lambda t}, \quad x \in \Lambda, \quad t \in \mathbb{R}.$$

Very strong properties can be deduced from the existence of such hyperbolic structure; see for instance [11,12,18,37,41].

More recently, extensions of this theory based on weaker notions of hyperbolicity, like the notions of dominated splitting, partial hyperbolicity, volume hyperbolicity and singular hyperbolicity (for three-dimensional flows) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see [7] and specifically [5,47] for singular hyperbolicity and Lorenz-like attractors.

One of the technical difficulties in this theory is to actually prove the existence of a hyperbolic structure, even in its weaker forms. We mention that Malkus showed that the Lorenz equations, presented in [22], are the equations of motion of a waterwheel, which was built at MIT in the 1970s and helped to convince the skeptical physicists of the reality of chaos; see [44, Section 9.1]. Only around the year 2000 was it established by Tucker in [45,46] that the Lorenz system of equations, with the parameters indicated by Lorenz, does indeed have a chaotic strange attractor. This proof is a computer assisted proof which works for a specific choice of parameters, and has not been improved to this day. More recently, Hunt and Mackay in [16] have shown that the behavior of a certain physical system, for a specific choice of parameters which can be fixed in a concrete laboratory setup, is modeled by an Anosov flow.

The most usual and geometric way to prove hyperbolicity is to use a field of cones. This idea goes as far back as the beginning of the hyperbolic theory; see Alekseev [1–3]. Given a continuous, splitting $T_\Lambda M = E \oplus F$ (not necessarily invariant with respect to a flow $X_t$) of the tangent space over an invariant subset $\Lambda$, a field of cones of size $a > 0$ centered around $F$ is defined by

$$C_a(x) := \{\vec{0}\} \cup \{(u, v) \in E_x \times F_x : \|u\| \le a\|v\|\}, \quad x \in \Lambda.$$

Let us assume that there exists $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ and every negative $t$

(1) $\overline{DX_t \cdot C_a(x)} \subset C_a(X_t(x))$ (the overline denotes closure in $T_{X_t(x)}M$);
(2) $\|DX_t \cdot w\| \le \lambda\|w\|$ for each $v \in C_a(x)$.

Then there exists an invariant bundle $E^s$ contained in the cone field $C_a$ over $\Lambda$ whose vectors are uniformly contracted. The complementary cone field satisfies the analogous to the first item above for positive $t$. This ensures the existence of a partially hyperbolic splitting over $\Lambda$.

We present a simple extension of the notion of infinitesimal Lyapunov function, from [17], to singular flows, and show how this technique provides a new characterization of partially hyperbolic structures for invariant sets for flows, and also of singular and sectional hyperbolicity. In the absence of singularities, we can also rephrase uniform hyperbolicity with the language of infinitesimal Lyapunov functions.

This technique is not new. Lewowicz used it in his study of expansive homeomorphisms [20] and Wojtkowski adapted it for the study of Lyapunov exponents in [49]. Using infinitesimal Lyapunov functions, Wojtkowski was able to show that the second item above is superfluous: the geometric condition expressed in the first item is actually enough to conclude uniform contraction.

Workers using these techniques, like Lewowicz [19], Markarian [25], Wojtkowski [51], Burns-Katok [17], have only considered either dynamical systems given by maps or by flows without singularities. In this last case, the authors deal with the linear Poincaré flow on the normal bundle to the flow direction.

We adapt ideas introduced first by Lewowicz in [19], and developed by several other authors in different contexts, to the setting of vector bundle automorphisms over flows with singularities; see also [25,50,52] for other known applications of this technique to billiards and symplectic flows, and also [34,35] for a general theory of $\mathcal{J}$-monotonous linear transformations. Recently in [6] a general framework was established relating Lyapunov exponents for invariant measures of maps and flows with eventually strict Lyapunov functions.

We improve on these results by showing, roughly, that the condition on item (1) above on a trapping region for a flow implies partial hyperbolicity, even when singularities are present. This also provides a way to define uniform hyperbolicity on a compact invariant set for a smooth flow generated by a vector field $X$, using only $X$ and $DX$ together with a family of Lyapunov functions.

We then provide an extra necessary and sufficient condition ensuring that the complementary cone, containing invariant subbundle $E^c$, which contains the flow direction, is such that the area form along any two-dimensional subspace of $E^c$ is uniformly expanded by the action of the tangent cocycle $DX_t$ of the flow $X_t$ (this property is today known as sectional-hyperbolicity; see [26]).

Moreover, these conditions can be expressed using the vector field $X$ and its space derivative $DX$ together with an infinitesimal Lyapunov function, and are reduced to checking that a certain symmetric operator is positive definite on all points of the trapping region. While we usually define hyperbolicity by using the differential $Df$ of a diffeomorphism $f$, or the cocycle $(DX_t)_{t\in\mathbb{R}}$ associated to the continuous one-parameter group $(X_t)_{t\in\mathbb{R}}$, we show how to express partial hyperbolicity using only the interplay between the infinitesimal generator $X$ of the group $X_t$, its derivative $DX$ and the infinitesimal Lyapunov function. Since in many situations dealing with mathematical models from the physical, engineering or social sciences, it is the vector field that is given and not the flow, we expect that the theory here presented to be useful to develop simpler algorithms to check hyperbolicity.

### 1.1 Preliminary definitions

Before the main statements we collect some definitions in order to state the main results.

Let $M$ be a connected compact finite $n$-dimensional manifold, $n \geq 3$, with or without boundary, together with a flow $X_t\colon M \to M$, $t \in \mathbb{R}$ generated by a $C^1$ vector field $X\colon M \to TM$, such that $X$ is inwardly transverse to the boundary $\partial M$, if $\partial M \neq \emptyset$.

An *invariant set* $\Lambda$ for the flow of $X$ is a subset of $M$ which satisfies $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. The *maximal invariant set of the flow* is $M(X) := \cap_{t\geq 0} X_t(M)$, which is clearly a compact invariant set.

A *trapping region* $U$ for a flow $X_t$ is an open subset of the manifold $M$ which satisfies: $X_t(U)$ is contained in $U$ for all $t > 0$; and there exists $T > 0$ such that $\overline{X_t(U)}$ is contained in the interior of $U$ for all $t > T$.

A *singularity* for the vector field $X$ is a point $\sigma \in M$ such that $X(\sigma) = \vec{0}$ or, equivalently, $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. The set formed by singularities is the *singular set of* $X$ denoted $\mathrm{Sing}(X)$. We say that a *singularity is hyperbolic* if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity $\sigma$ have nonzero real part.

**Definition 1** A *dominated splitting* over a compact invariant set $\Lambda$ of $X$ is a continuous $DX_t$-invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants $K, \lambda$ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \quad \text{for all } x \in \Lambda, \text{ and all } t > 0. \tag{1.1}$$

A compact invariant set $\Lambda$ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle $E$ is uniformly contracted. In this case $F$ is the *central subbundle* of $\Lambda$.

A compact invariant set $\Lambda$ is said to be *singular hyperbolic* if it is partially hyperbolic and the action of the tangent cocycle expands volume along the central subbundle, i.e.,

$$|\det(DX_t|_{F_x})| > Ce^{\lambda t}, \quad \forall t > 0, \ \forall \, x \in \Lambda. \tag{1.2}$$

**Definition 2** We say that a $DX_t$-invariant subbundle $F \subset T_\Lambda M$ is a *sectionally expanding* subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants $C, \lambda$ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \quad \forall t > 0. \tag{1.3}$$

And, finally, the definition of sectional-hyperbolicity.

**Definition 3** [26, Definition 2.7] A *sectional hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and the central subbundle is sectionally expanding.

We note that a sectional hyperbolic set always is singular hyperbolic, however the reverse is only true in dimension three, not in higher dimensions; see for instance Metzger–Morales [26] and Zu–Gan–Wen [54].

*Remark 1.1* The properties of sectional hyperbolicity can be expressed in the following equivalent way; see [5]. There exists $T > 0$ such that

- $\|DX_T|_{E_x}\| < \frac{1}{2}$ for all $x \in \Lambda$ (uniform contraction); and
- $|\det(DX_T|_{F_x})| > 2$ for all $x \in \Lambda$ and each 2-subspace $F_x$ of $E^c_x$ (2-sectional expansion).

We say that a compact invariant set $\Lambda$ is a *volume hyperbolic* set if it has a dominated splitting $E \oplus F$ such that the volume along its subbundles is uniformly contracted (along $E$) and expanded (along $F$) by the action of the tangent cocyle. If the whole manifold $M$ is a volume hyperbolic set for a flow $X_t$, then we say that $X_t$ is a volume-hyperbolic flow.

We recall that a flow $X_t$ is said to be *Anosov* if the whole manifold $M$ is a hyperbolic set for the flow. Based on this definition, we say that $X_t$ is a *sectional Anosov flow* if the maximal invariant set $M(X)$ is a sectional-hyperbolic set for the flow.

From now on, we consider $M$ a connected compact finite dimensional Riemannian manifold, $U \subset M$ a trapping region, $\Lambda(U) = \Lambda_X(U) := \cap_{t>0}\overline{X_t(U)}$ the maximal positive invariant subset in $U$ for the vector field $X$ and $E_U$ a finite dimensional vector bundle over $U$. We also assume that all singularities of $X$ in $U$ (if they exist) are hyperbolic.

### *1.1.1 Fields of quadratic forms: positive and negative cones*

Let $E_U$ be a finite dimensional vector bundle with base $U$ and $\mathcal{J}: E_U \to \mathbb{R}$ be a continuous field of quadratic forms $\mathcal{J}_x: E_x \to \mathbb{R}$ which are non-degenerate and have index $0 < q < \dim(E) = n$. The index $q$ of $\mathcal{J}$ means that the maximal dimension of subspaces of non-positive vectors is $q$.

We also assume that $(\mathcal{J}_x)_{x\in U}$ is continuously differentiable along the flow. The continuity assumption on $\mathcal{J}$ means that for every continuous section $Z$ of $E_U$ the map $U \ni x \mapsto \mathcal{J}(Z(x)) \in \mathbb{R}$ is continuous. The $C^1$ assumption on $\mathcal{J}$ along the flow means that the map $\mathbb{R} \ni t \mapsto \mathcal{J}_{X_t(x)}(Z(X_t(x))) \in \mathbb{R}$ is continuously differentiable for all $x \in U$ and each $C^1$ section $Z$ of $E_U$.

We let $\mathcal{C}_\pm = \{C_\pm(x)\}_{x\in U}$ be the field of positive and negative cones associated to $\mathcal{J}$

$$C_\pm(x) := \{0\} \cup \{v \in E_x : \pm\,\mathcal{J}_x(v) > 0\} \quad x \in U$$

and also let $\mathcal{C}_0 = \{C_0(x)\}_{x\in U}$ be the corresponding field of zero vectors $C_0(x) = \mathcal{J}_x^{-1}(\{0\})$ for all $x \in U$.

### *1.1.2 Linear multiplicative cocycles over flows*

Let $A: E \times \mathbb{R} \to E$ be a smooth map given by a collection of linear bijections

$$A_t(x): E_x \to E_{X_t(x)}, \quad x \in M, t \in \mathbb{R},$$

where $M$ is the base space of the finite dimensional vector bundle $E$, satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in M, t, s \in \mathbb{R},$$

with $\{X_t\}_{t\in\mathbb{R}}$ a smooth flow over $M$. We note that for each fixed $t > 0$ the map $A_t: E \to E$, $v_x \in E_x \mapsto A_t(x) \cdot v_x \in E_{X_t(x)}$ is an automorphism of the vector bundle $E$.

The natural example of a linear multiplicative cocycle over a smooth flow $X_t$ on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle $TM$ of a finite dimensional compact manifold $M$.

The following definitions are fundamental to state our results.

**Definition 4** Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$ with constant index on the trapping region $U$ for the flow $X_t$, we say that the cocycle $A_t(x)$ over $X$ is

- $\mathcal{J}$-*separated* if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- *strictly* $\mathcal{J}$-*separated* if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- $\mathcal{J}$-*monotone* if $\mathcal{J}_{X_t(x)}(A_t(x)v) \geq \mathcal{J}_x(v)$, for each $v \in T_xM\setminus\{\vec{0}\}$ and $t > 0$;
- *strictly* $\mathcal{J}$-*monotone* if $\partial_t\big(\mathcal{J}_{X_t(x)}(A_t(x)v)\big)\mid_{t=0} > 0$, for all $v \in T_xM\setminus\{0\}$, $t > 0$ and $x \in U$;
- $\mathcal{J}$-*isometry* if $\mathcal{J}_{X_t(x)}(A_t(x)v) = \mathcal{J}_x(v)$, for each $v \in T_xM$ and $x \in U$.

Thus, $\mathcal{J}$-separation corresponds to simple cone invariance and strict $\mathcal{J}$-separation corresponds to strict cone invariance under the action of $A_t(x)$.

We say that the flow $X_t$ is (strictly) $\mathcal{J}$-*separated* on $U$ if $DX_t(x)$ is (strictly) $\mathcal{J}$-*separated* on $T_UM$. Analogously, the flow of $X$ on $U$ is (strictly) $\mathcal{J}$-*monotone* if $DX_t(x)$ is (strictly) $\mathcal{J}$-*monotone*.

*Remark 1.2* If a flow is strictly $\mathcal{J}$-separated, then for $v \in T_x M$ such that $\mathcal{J}_x(v) \le 0$ we have $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) < 0$ for all $t > 0$ and $x$ such that $X_{-s}(x) \in U$ for every $s \in [-t, 0]$. Indeed, otherwise $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) \ge 0$ would imply $\mathcal{J}_x(v) = \mathcal{J}_x\big(DX_t(DX_{-t}(v))\big) > 0$, contradicting the assumption that $v$ was a non-positive vector.

This means that a flow $X_t$ is strictly $\mathcal{J}$-separated if, and only if, its time reversal $X_{-t}$ is strictly $(-\mathcal{J})$-separated.

A vector field $X$ is $\mathcal{J}$-*non-negative* on $U$ if $\mathcal{J}(X(x)) \ge 0$ for all $x \in U$, and $\mathcal{J}$-*non-positive* on $U$ if $\mathcal{J}(X(x)) \le 0$ for all $x \in U$. When the quadratic form used in the context is clear, we will simply say that $X$ is non-negative or non-positive.

## 1.2 Statement of the results

We say that a compact invariant subset $\Lambda$ is *non-trivial* if

- either $\Lambda$ does not contain singularities;
- or $\Lambda$ contains at most finitely many singularities, $\Lambda$ contains some regular orbit and is connected.

Our main result is the following.

**Theorem A** *A non-trivial attracting set $\Lambda$ of a trapping region $U$ is partially hyperbolic for a flow $X_t$ if, and only if, there is a $C^1$ field of non-degenerate quadratic forms $\mathcal{J}$ with constant index, equal to the dimension of the stable subspace of $\Lambda$, such that $X_t$ is non-negative strictly $\mathcal{J}$-separated on $U$.*

This is a direct consequence of a corresponding result for linear multiplicative cocycles over vector bundles which we state in Theorem 2.13 and prove in Sect. 2.4.

We obtain a criterion for partial and uniform hyperbolicity which extends the one given by Lewowicz [19] and Wojtkowski [52]. The condition of strict $\mathcal{J}$-separation can be expressed only using the vector field $X$ and its spatial derivative $DX$, as follows.

**Proposition 1.3** *A $\mathcal{J}$-non-negative vector field $X$ on $U$ is (strictly) $\mathcal{J}$-separated if, and only if, there exists a compatible field of forms $\mathcal{J}_0$ and there exists a function $\delta\colon U \to \mathbb{R}$ such that the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^* \cdot J_0$ satisfies*

$$\tilde{J}_{0,x} - \delta(x) J_0 \quad \textit{is positive (definite) semidefinite}, \quad x \in U,$$

*where $DX(x)^*$ is the adjoint of $DX(x)$ with respect to the adapted inner product.*

In the statement above, we say that a field of quadratic forms $\mathcal{J}_0$ on $U$ is *compatible* to $\mathcal{J}$, and we write $\mathcal{J} \sim \mathcal{J}_0$, if there exists $C > 1$ satisfying for $x \in \Lambda$

$$\frac{1}{C} \cdot \mathcal{J}_0(v) \le \mathcal{J}(v) \le C \cdot \mathcal{J}_0(v), \quad v \in E_x \cup F_x,$$

where $E \oplus F$ is a $DX_t$-invariant splitting of $T_\Lambda M$.

Again this is a consequence of a corresponding result for linear multiplicative cocycles where $DX$ is replaced by the infinitesimal generator

$$D(x) := \lim_{t \to 0} \frac{A_t(x) - Id}{t}$$

of the cocycle $A_t(x)$.

As a consequence of Theorem A, we characterize hyperbolic maximal invariant subsets in trapping regions without singularities as follows. We recall that the index of a (partially) hyperbolic set is the dimension of the uniformly contracted subbundle of its tangent bundle.

**Corollary B** *The maximal invariant subset $\Lambda$ of $U$ is a hyperbolic set for $X$ of index $s$ if, and only if, there exist $\mathcal{J}, \mathcal{G}$ smooth fields of non-degenerate quadratic forms on $U$ with constant index $s$ and $n-s-1$, respectively, where $s < n-2$ and $n = dim(M)$, such that $X_t$ is strictly $\mathcal{J}$-separated non-negative on $U$ with respect to $\mathcal{J}$, $X_t$ is strictly $\mathcal{G}$-separated non-positive with respect to $\mathcal{G}$, and there are no singularities of $X$ in $U$.*

### 1.2.1 Incompressible vector fields

To state the next result, we recall that a vector field is said to be *incompressible* if its flow has null divergence, i.e., it is volume-preserving on $M$.

In this particular case, we have the following easy corollary of Theorem A, since a partially hyperbolic flow in a compact manifold must expand volume along the central direction. Moreover, if the stable direction has codimension 2, the central direction expands area.

**Corollary C** *Let $X$ be a $C^1$ incompressible vector field on a compact finite dimensional manifold $M$ which is non-negative and strictly $\mathcal{J}$-separated for a field of non-degenerate and indefinite quadratic forms $\mathcal{J}$ with index* $\operatorname{ind}(\mathcal{J}) = \dim(M) - 2$. *Then $X_t$ is an Anosov flow.*

Indeed, the results of Doering [13] and Morales–Pacifico–Pujals [29], in dimension three, Vivier [48] and Li–Gan–Wen [21], in higher dimensions, ensure that there are no singularities in the interior of a sectional-hyperbolic set, and so this set is hyperbolic; see Sect. 3.1.

To present the results about sectional-hyperbolicity, we need some more definitions.

### 1.2.2 $\mathcal{J}$-monotonous linear Poincaré flow

We apply these notions to the linear Poincaré flow defined on regular orbits of $X_t$ as follows.

We assume that the vector field $X$ is non-negative on $U$. Then, the span $E_x^X$ of $X(x) \neq \vec{0}$ is a $\mathcal{J}$-non-degenerate subspace. According to item (1) of Proposition 2.1, this means that $T_x M = E_x^X \oplus N_x$, where $N_x$ is the pseudo-orthogonal complement of $E_x^X$ with respect to the bilinear form $\mathcal{J}$, and $N_x$ is also non-degenerate. Moreover, by the definition, the index of $\mathcal{J}$ restricted to $N_x$ is the same as the index of $\mathcal{J}$. Thus, we can define on $N_x$ the cones of positive and negative vectors, respectively, $N_x^+$ and $N_x^-$, just like before.

Now we define the linear Poincaré flow $P^t$ of $X_t$ along the orbit of $x$, by projecting $DX_t$ orthogonally (with respect to $\mathcal{J}$) over $N_{X_t(x)}$ for each $t \in \mathbb{R}$:

$$P^t v := \Pi_{X_t(x)} DX_t v, \quad v \in T_x M, t \in \mathbb{R}, X(x) \neq \vec{0},$$

where $\Pi_{X_t(x)}: T_{X_t(x)}M \to N_{X_t(x)}$ is the projection on $N_{X_t(x)}$ parallel to $X(X_t(x))$. We remark that the definition of $\Pi_x$ depends on $X(x)$ and $\mathcal{J}_X$ only. The linear Poincaré flow $P^t$ is a linear multiplicative cocycle over $X_t$ on the set $U$ with the exclusion of the singularities of $X$.

In this setting we can say that the linear Poincaré flow is (strictly) $\mathcal{J}$-separated and (strictly) $\mathcal{J}$-monotonous using the non-degenerate bilinear form $\mathcal{J}$ restricted to $N_x$ for a regular $x \in U$. More precisely: $P^t$ is $\mathcal{J}$-monotonous if $\partial_t \mathcal{J}(P^t v) \mid_{t=0} \geq 0$, for each $x \in U, v \in T_x M \setminus \{\vec{0}\}$ and $t > 0$, and strictly $\mathcal{J}$-monotonous if $\partial_t \mathcal{J}(P^t v) \mid_{t=0} > 0$, for all $v \in T_x M \setminus \{\vec{0}\}$, $t > 0$ and $x \in U$.

**Theorem D** *Let $\Lambda$ be a non-trivial attracting set $\Lambda$ of $U$ which is contained in the non-wandering set $\Omega(X)$. Then $\Lambda$ is sectional hyperbolic for $X_t$ if, and only if, there is a $C^1$ field $\mathcal{J}$ of non-degenerate quadratic forms with constant index, equal to the dimension of* *the stable subspace of* $\Lambda$*, such that* $X_t$ *is a non-negative strictly* $\mathcal{J}$*-separated flow on* $U$*, whose singularities are sectionally hyperbolic with index* $\operatorname{ind}(\sigma) \geq \operatorname{ind}(\Lambda)$*, and for each compact invariant subset* $\Gamma$ *of* $\Lambda$ *without singularities there exists a field of quadratic forms* $\mathcal{J}_0$ *equivalent to* $\mathcal{J}$ *so that the linear Poincaré flow is strictly* $\mathcal{J}_0$*-monotonous on* $\Gamma$.

As usual, we say that $q \in M$ is *non-wandering* for $X$ if for every $T > 0$ and every neighborhood $W$ of $q$ there is $t > T$ such that $X_t(W) \cap W \neq \emptyset$. The set of non-wandering points of $X$ is denoted by $\Omega(X)$. A singularity $\sigma$ is *sectionally hyperbolic with index* $\operatorname{ind}(\sigma)$ if $\sigma$ is a hyperbolic singularity with stable direction $E^s_\sigma$ having dimension $\operatorname{ind}(\sigma)$ and a central direction $E^c_\sigma$ such that $T_\sigma M = E^s_\sigma \oplus E^c_\sigma$ is a $DX_t(\sigma)$-invariant splitting, $E^s_\sigma$ is uniformly contracted and $E^c_\sigma$ is sectionally expanded by the action of $DX_t(\sigma)$.

As before, the condition of $\mathcal{J}$-monotonicity for the linear Poincaré flow can be expressed using only the vector field $X$ and its space derivative $DX$ as follows.

**Proposition 1.4** *A* $\mathcal{J}$*-non-negative vector field* $X$ *on a forward invariant region* $U$ *has a linear Poincaré flow which is (strictly)* $\mathcal{J}$*-monotone if, and only if, the operator* $\hat{J}_x := DX(x)^* \cdot \Pi_x^* J \Pi_x + \Pi_x^* J \Pi_x \cdot DX(x)$ *is a (positive) non-negative self-adjoint operator, that is, all eigenvalues are (positive) non-negative, for each* $x \in U$ *such that* $X(x) \neq \vec{0}$.

The conditions above are again consequence of the corresponding results for linear multiplicative cocycles over flows, as explained in Sect. 4.

### 1.3 Examples

With the equivalence provided by Theorems A and D and Corollaries B and C, we have plenty of examples illustrating our results.

*Example 1* We can consider

- the classical examples of uniformly hyperbolic attractors for $C^1$ flows, in any dimension greater or equal to 3; see e.g. [12].
- The classical Lorenz attractor from the Lorenz ODE system and the geometrical Lorenz attractors; see e.g. [22,45,47].
- Singular-hyperbolic (or Lorenz-like) attractors and attracting sets in three dimensions; see e.g. [5,28,30].
- Contracting Lorenz (Rovella) attractors; see e.g. [27,38].
- Sectional-hyperbolic attractors for dimensions higher than three; see e.g [7,9,26].
- Multidimensional Rovella-like attractors; see [4].

The following examples illustrate the fact that the change of coordinates to adapt the quadratic forms as explained in Sect. 2 is important in applications.

*Example 2* Given a diffeomorphism $f: \mathbb{T}^2 \to \mathbb{T}^2$ of the 2-torus, let $X_t: M \to M$ be a suspension flow with roof function $r: M \to [r_0, r_1]$ over the base transformation $f$, where $0 < r_0 < r_1$ are fixed, as follows.

We define $M := \{(x, y) \in \mathbb{T}^2 \times [0, +\infty): 0 \leq y < r(x)\}$. For $x = x_0 \in \mathbb{T}^2$ we denote by $x_n$ the $n$th iterate $f^n(x_0)$ for $n \geq 0$ and by $S_n\varphi(x_0) = S_n^f \varphi(x_0) = \sum_{j=0}^{n-1} \varphi(x_j)$ the $n$th-ergodic sum, for $n \geq 1$ and for any given real function $\varphi: \mathbb{T}^2 \to \mathbb{R}$ in what follows. Then for each pair $(x_0, s_0) \in M$ and $t > 0$ there exists a unique $n \geq 1$ such that $S_n r(x_0) \leq s_0 + t < S_{n+1} r(x_0)$ and we define

$$X_t(x_0, s_0) = \big(x_n, s_0 + t - S_n r(x_0)\big).$$

We note that the vector field corresponding to this suspension flow is the constant vector field $X = (0, 1)$. We observe that the space $M$ becomes a compact manifold if we identify $(x, r(x))$ with $(f(x), 0)$; see e.g. [32].

Hence, if we are given a field of quadratic forms $\mathcal{J}$ on $M$ and do not change coordinates accordingly, we obtain $DX \equiv 0$ and so the relation provided by Proposition 1.3 will not be fulfilled, because $\tilde{J}_x - \delta(x)J = -\delta(x)J$ is not positive definite for any choice of $\delta$.

*Remark 1.5* Moreover, if a flow $X_t$ is such that the infinitesimal generator $X$ is constant in the ambient space is $\mathcal{J}$-separated, then strict $\mathcal{J}$-separation implies that $-\delta(x)J$ is positive definite for all $x \in U$, and so $\delta$ is the null function on the trapping region.

*Example 3* Now consider the same example as above but now $f$ is an Anosov diffeomorphism of $\mathbb{T}^2$ with the hyperbolic splitting $E^s \oplus E^u$ defined at every point. Then the semiflow will be partially hyperbolic with splitting $E^s \oplus (E^X \oplus E^u)$ where $E^X$ is the one-dimensional bundle spanned by the flow direction: $E^X_{(x,s)} = \mathbb{R} \cdot X(x, s)$, $(x, s) \in M$.

Hence, Theorem A ensures the existence of a field $\mathcal{J}$ of quadratic forms such that $X^t$ is strictly $\mathcal{J}$-separated.

Comparing with the observation at the end of Example 2, this demands a change of coordinates and, in those coordinates, the vector field $X$ will no longer be a constant vector field.

*Example 4* Now we present a suspension flow whose base map has a dominated splitting but the flow does not admit any dominated splitting.

Let $f: \mathbb{T}^4 \times \mathbb{T}^4$ be the diffeomorphism described in [10] which admits a continuous dominated splitting $E^{cs} \oplus E^{cu}$ on $\mathbb{T}^4$, but does not admit any hyperbolic (uniformly contracting or expanding) subbundle. There are hyperbolic fixed points of $f$ satisfying, see Fig. 1:

- $\dim E^u(p) = 2 = \dim E^s(p)$ and there exists no invariant one-dimensional subbundle of $E^u(p)$;
- $\dim E^u(\widetilde{p}) = 2 = \dim E^s(\widetilde{p})$ and there exists no invariant one-dimensional subbundle of $E^s(\widetilde{p})$;
- $\dim E^s(\widetilde{q}) = 3$ and $\dim E^u(q) = 3$.

Hence, the suspension semiflow of $f$ with constant roof function 1 does not admit any dominated splitting. In fact, the natural invariant splitting $E^{cs} \oplus E^X \oplus E^{cu}$ is the continuous invariant splitting over $\mathbb{T}^4 \times [0, 1]$ with bundles of least dimension, and is not dominated since at the point $p$ the flow direction $E^X(p)$ dominates the $E^{cs}(p) = E^s(p)$ direction, but at the point $q$ this domination is impossible.

We now present simple cases of partial hyperbolicity/strict separation and hyperbolicity/strict monotonicity, obtaining explicitly the function $\delta$.

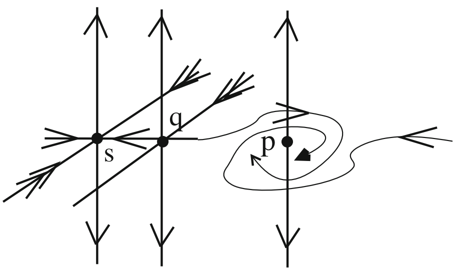


**Fig. 1** Saddles with real and complex eigenvalues

*Example 5* Let us consider a hyperbolic saddle singularity $\sigma$ at the origin for a smooth vector field $X$ on $\mathbb{R}^3$ such that the eigenvalues of $DX(\sigma)$ are real and satisfy $\lambda_1 < \lambda_2 < 0 < \lambda_3$. Through a coordinate change, we may assume that $DX(\sigma) = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. We consider the following quadratic forms in $\mathbb{R}^3$.

**Index 1**: $\mathcal{J}_1(x, y, z) = -x^2 + y^2 + z^2$. Then $\mathcal{J}_1$ is represented by the matrix $J_1 = \mathrm{diag}\{-1, 1, 1\}$, that is, $\mathcal{J}_1(\vec{w}) = \langle J_1(\vec{w}), \vec{w}\rangle$ with the canonical inner product.
Then $\tilde{J}_1 = J_1 \cdot DX(\sigma) + DX(\sigma)^* \cdot J_1 = \mathrm{diag}\{-2\lambda_1, 2\lambda_2, 2\lambda_3\}$ and $\tilde{J}_1' - \delta J_1 > 0 \iff 2\lambda_1 < \delta < 2\lambda_2 < 0$. So $\delta$ must be negative and $\tilde{J}_1$ is not positive definite. From Proposition 1.3 and Theorem 2.7 we have strict $\mathcal{J}_1$-separation, thus partial hyperbolicity with the negative $x$-axis a uniformly contracted direction dominated by the $yz$-direction; but this is not an hyperbolic splitting.
Moreover the conclusion would be the same if $\lambda_3$ where negative: we get a sink with a partially hyperbolic splitting.
**Index 2**: $\mathcal{J}_2(x, y, z) = -x^2 - y^2 + z^2$ represented by $J_2 = \mathrm{diag}\{-1, -1, 1\}$.
Then $\tilde{J}_2 = \mathrm{diag}\{-2\lambda_1, -2\lambda_2, 2\lambda_3\}$ and $\tilde{J}_2 - \delta J_1 > 0 \iff 2\lambda_2 < \delta < 2\lambda_3$. So $\delta$ might be either positive or negative, but we still have strict $\mathcal{J}_2$-separation, and $\tilde{J}_2$ is positive definite. Hence by Theorem 2.7 $X_t$ is strictly $\mathcal{J}_2$-monotone at $\sigma$ and the splitting $\mathbb{R}^3 = (\mathbb{R}^2 \times \{0\}) \oplus (\{(0, 0)\} \times \mathbb{R})$ is hyperbolic.
**Index 1, not separated**: $\mathcal{J}_3(x, y, z) = x^2 - y^2 + z^2$ represented by $J_2 = \mathrm{diag}\{1, -1, 1\}$. Now $\tilde{J}_3 = \mathrm{diag}\{2\lambda_1, -2\lambda_2, 2\lambda_3\}$ is not positive definite and $\tilde{J}_3 - \delta J_3$ is the diagonal matrix $\mathrm{diag}\{2\lambda_1-\delta, -2\lambda_2+\delta, 2\lambda_3-\delta\}$ which represents a positive semidefinite quadratic form if, and only if, $\delta \le 2\lambda_1$, $\delta \ge 2\lambda_2$ and $\delta \le 2\lambda_3$, which is impossible. Hence we do not have domination of the $y$-axis by the $xz$-direction.

### 1.4 Organization of the text

We study $\mathcal{J}$-separated linear multiplicative cocycles over flows in Sect. 2.2, where we prove the main results whose specialization for the derivative cocycle of a smooth flow provide the main theorems, including Proposition 1.3. We then consider the case of the derivative cocycle and prove Theorem A and Corollaries B and C in Sect. 3. We turn to study sectional hyperbolic attracting sets in Sect. 4, where we prove Theorem D and Proposition 1.4.

## 2 Some properties of quadratic forms and $\mathcal{J}$-separated cocycles

The assumption that $M$ is a compact manifold enables us to globally define an inner product in $E$ with respect to which we can find the an orthonormal basis associated to $\mathcal{J}_x$ for each $x$, as follows. Fixing an orthonormal basis on $E_x$ we can define the linear operator

$$J_x : E_x \to E_x \quad \text{such that} \quad \mathcal{J}_x(v) = \langle J_x v, v\rangle \qquad \text{for all} \quad v \in T_x M,$$

where $\langle , \rangle = \langle , \rangle_x$ is the inner product at $E_x$. Since we can always replace $J_x$ by $(J_x + J_x^*)/2$ without changing the last identity, where $J_x^*$ is the adjoint of $J_x$ with respect to $\langle , \rangle$, we can assume that $J_x$ is self-adjoint without loss of generality. Hence, we represent $\mathcal{J}(v)$ by a non-degenerate symmetric bilinear form $\langle J_x v, v\rangle_x$.

### 2.1 Adapted coordinates for the quadratic form

Now we use Lagrange's method to diagonalize this bilinear form, obtaining a base $\{u_1, \dots, u_n\}$ of $E_x$ such that

$$\mathcal{J}_x\left(\sum_i \alpha_i u_i\right) = \sum_{i=1}^{q} -\lambda_i \alpha_i^2 + \sum_{j=q+1}^{n} \lambda_j \alpha_j^2, \quad (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n.$$

Replacing each element of this base according to $v_i = |\lambda_i|^{1/2} u_i$ we deduce that

$$\mathcal{J}_x\left(\sum_i \alpha_i v_i\right) = \sum_{i=1}^{q} -\alpha_i^2 + \sum_{j=q+1}^{n} \alpha_j^2, \quad (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n.$$

Finally, we can redefine $\langle , \rangle$ so that the base $\{v_1, \ldots, v_n\}$ is orthonormal. This can be done smoothly in a neighborhood of $x$ in $M$ since we are assuming that the quadratic forms are non-degenerate; the reader can check the method of Lagrange in a standard Linear Algebra textbook and observe that the steps can be performed robustly and smoothly for all nearby tangent spaces; see for instance [33,40].

In this adapted inner product we have that $J_x$ has entries from $\{-1, 0, 1\}$ only, $J_x^* = J_x$ and also that $J_x^2 = J_x$. Having fixed the orthonormal frame as above, the *standard negative subspace* at $x$ is the one spanned by $v_1, \ldots, v_q$ and the *standard positive subspace* at $x$ is the one spanned $v_{q+1}, \ldots, v_n$.

### *2.1.1 $\mathcal{J}$-symmetrical matrixes and $\mathcal{J}$-selfadjoint operators*

The symmetrical bilinear form defined by $(v, w) = \langle J_x v, w \rangle$, $v, w \in E_x$ for $x \in M$ endows $E_x$ with a pseudo-Euclidean structure. Since $\mathcal{J}_x$ is non-degenerate, then the form $(\cdot, \cdot)$ is likewise non-degenerate and many properties of inner products are shared with symmetrical non-degenerate bilinear forms. We state some of them below.

**Proposition 2.1** *Let $(\cdot, \cdot): V \times V \to \mathbb{R}$ be a real symmetric non-degenerate bilinear form on the real finite dimensional vector space $V$.*

(1) *$E$ is a subspace of $V$ for which $(\cdot, \cdot)$ is non-degenerate if, and only if, $V = E \oplus E^{\perp}$. We recall that $E^{\perp} := \{v \in V : (v, w) = 0 \text{ for all } w \in E\}$, the pseudo-orthogonal space of $E$, is defined using the bilinear form.*
(2) *Every base $\{v_1, \ldots, v_n\}$ of $V$ can be orthogonalized by the usual Gram–Schmidt process of Euclidean spaces, that is, there are linear combinations of the basis vectors $\{w_1, \ldots, w_n\}$ such that they form a basis of $V$ and $(w_i, w_j) = 0$ for $i \neq j$. Then this last base can be pseudo-normalized: letting $u_i = |(w_i, w_i)|^{-1/2} w_i$ we get $(u_i, u_j) = \pm\delta_{ij}$, $i, j = 1, \ldots, n$.*
(3) *There exists a maximal dimension $p$ for a subspace $P_+$ of $\mathcal{J}$-positive vectors and a maximal dimension $q$ for a subspace $P_-$ of $\mathcal{J}$-negative vectors; we have $p + q = \dim V$ and $q$ is known as the* index *of $\mathcal{J}$.*
(4) *For every linear map $L: V \to \mathbb{R}$ there exists a unique $v \in V$ such that $L(w) = (v, w)$ for each $w \in V$.*
(5) *For each $L: V \to V$ linear there exists a unique linear operator $L^+: V \to V$ (the pseudo-adjoint) such that $(L(v), w) = (v, L^+(w))$ for every $v, w \in V$.*
(6) *Every pseudo-self-adjoint $L: V \to V$, that is, such that $L = L^+$, satisfies*
    (a) *eigenspaces corresponding to distinct eigenvalues are pseudo-orthogonal;*
    (b) *if a subspace $E$ is $L$-invariant, then $E^{\perp}$ is also $L$-invariant.*

The proofs are rather standard and can be found in [23].

### 2.2 Properties of $\mathcal{J}$-separated cocycles

In what follows we usually drop the subscript indicating the point where $\mathcal{J}$ is calculated to avoid heavy notation, since the base point is clear from the context.

#### *2.2.1 $\mathcal{J}$-separated linear maps*

The following simple result will be very useful in what follows.

**Lemma 2.2** *Let $V$ be a real finite dimensional vector space endowed with a non-positive definite and non-degenerate quadratic form $\mathcal{J}: V \to \mathbb{R}$.*

*If a symmetric bilinear form $F: V \times V \to \mathbb{R}$ is non-negative on $C_0$ then*

$$r_+ = \inf_{v \in C_+} \frac{F(v,v)}{\langle Jv, v\rangle} \geq \sup_{u \in C_-} \frac{F(u,u)}{\langle Ju, u\rangle} = r_-$$

*and for every $r$ in $[r_-, r_+]$ we have $F(v,v) \geq r\langle Jv, v\rangle$ for each vector $v$.*

*In addition, if $F(\cdot,\cdot)$ is positive on $C_0 \setminus \{\vec{0}\}$, then $r_- < r_+$ and $F(v,v) > r\langle Jv, v\rangle$ for all vectors $v$ and $r \in (r_-, r_+)$.*

*Proof* This can be found in [52] and also in [35]. We present the simple proof here for completeness.

Let us assume that the $F$ is non-negative on $C_0$ and argue by contradiction: we also assume that

$$\inf_{v \in C_+} \frac{F(v,v)}{\langle Jv, v\rangle} < \sup_{u \in C_-} \frac{F(u,u)}{\langle Ju, u\rangle}. \tag{2.1}$$

Hence we can find $v_0 \in C+$ and $u_0 \in C_-$ with $\mathcal{J}(v_0) = 1$ and $\mathcal{J}(u_0) = -1$ such that $F(v_0, v_0) + F(u_0, u_0) < 0$. We can also find an angle $\alpha$ such that both linear combinations

$$v = v_0 \cos\alpha + u_0 \sin\alpha \quad \text{and} \quad w = -v_0 \sin\alpha + u_0 \cos\alpha$$

belong to $C_0$. Then we must have $F(v,v) \geq 0$ and $F(w,w) \geq 0$, but we also have

$$\begin{aligned} F(v,v) + F(w,w) &= \cos^2\alpha \cdot F(v_0,v_0) + + \sin 2\alpha \cdot F(u_0,v_0) + \sin^2\alpha \cdot F(u_0,u_0) \\ &\quad + \sin^2\alpha \cdot F(v_0,v_0) - \sin 2\alpha \cdot F(u_0,v_0) + \cos^2\alpha \cdot F(u_0,u_0) \\ &= F(v_0,v_0) + F(u_0,u_0) < 0 \end{aligned}$$

and this contradiction shows that the opposite of (2.1) must be true.

Analogously, if $F$ is positive on $C_0 \setminus \{\vec{0}\}$, then we can argue in the same way: we assume that (2.1) is true with $\leq$ in the place of $<$; we obtain $F(v_0,v_0)+F(u_0,u_0) \leq 0$ and then construct $v, w$ such that $F(v,v) + F(w,w) > 0$; and, finally, we show that $F(v,v) + F(w,w) = F(v_0,v_0) + F(u_0,u_0) \leq 0$ to arrive again at a contradiction. □

*Remark 2.3* Lemma 2.2 shows that if $F(v,w) = \langle \tilde{J}v, w\rangle$ for some self-adjoint operator $\tilde{J}$ and $F(v,v) \geq 0$ for all $v$ such that $\langle Jv, v\rangle = 0$, then we can find $a \in \mathbb{R}$ such that $\tilde{J} \geq aJ$. This means precisely that $\langle \tilde{J}v, v\rangle \geq a\langle Jv, v\rangle$ for all $v$.

If, in addition, we have $F(v,v) > 0$ for all $v$ such that $\langle Jv, v\rangle = 0$, then we obtain a strict inequality $\tilde{J} > aJ$ for some $a \in \mathbb{R}$ since the infimum in the statement of Lemma 2.2 is strictly bigger than the supremum.

The (longer) proofs of the following results can be found in [52] or in [35]; see also [53].

**Proposition 2.4** *Let $L\colon V \to V$ be a $\mathcal{J}$-separated linear operator. Then*

(1) *$L$ can be uniquely represented by $L = RU$, where $U$ is a $\mathcal{J}$-isometry (i.e. $\mathcal{J}(U(v)) = \mathcal{J}(v)$, $v \in V$) and $R$ is $\mathcal{J}$-symmetric (or $\mathcal{J}$-pseudo-adjoint; see Proposition 2.1) with positive spectrum.*

(2) *the operator $R$ can be diagonalized by a $\mathcal{J}$-isometry. Moreover the eigenvalues of $R$ satisfy*

$$0 < r_-^q \le \cdots \le r_-^1 = r_- \le r_+ = r_1^+ \le \cdots \le r_+^p.$$

(3) *the operator $L$ is (strictly) $\mathcal{J}$-monotonous if, and only if, $r_- \le (<)1$ and $r_+ \ge (>)1$.*

For a $\mathcal{J}$-separated operator $L\colon V \to V$ and a $d$-dimensional subspace $F_+ \subset C_+$, the subspaces $F_+$ and $L(F_+) \subset C_+$ have an inner product given by $\mathcal{J}$. Thus both subspaces are endowed with volume elements. Let $\alpha_d(L; F_+)$ be the rate of expansion of volume of $L\mid_{F_+}$ and $\sigma_d(L)$ be the infimum of $\alpha_d(L; F_+)$ over all $d$-dimensional subspaces $F_+$ of $C_+$.

**Proposition 2.5** *We have $\sigma_d(L) = r_+^1 \dots r_+^d$, where $r_+^i$ are given by Proposition 2.4(2). Moreover, if $L_1, L_2$ are $\mathcal{J}$-separated, then $\sigma_d(L_1L_2) \ge \sigma_d(L_1)\sigma_d(L_2)$.*

The following corollary is very useful.

**Corollary 2.6** *For $\mathcal{J}$-separated operators $L_1, L_2\colon V \to V$ we have*

$$r_+^1(L_1L_2) \ge r_+^1(L_1)r_+^1(L_2) \quad \textit{and} \quad r_-^1(L_1L_2) \le r_-^1(L_1)r_-^1(L_2).$$

*Moreover, if the operators are strictly $\mathcal{J}$-separated, then the inequalities are strict.*

### 2.3 $\mathcal{J}$-separated linear cocycles over flows

The results in the previous subsection provide the following characterization of $\mathcal{J}$-separated cocycles $A_t(x)$ over a flow $X_t$ in terms of the infinitesimal generator $D(x)$ of $A_t(x)$; see (2.2). The following statement is more precise than Proposition 1.3.

Let $A_t(x)$ a linear multiplicative cocycles over a flow $X_t$. We define the infinitesimal generator of $A_t(x)$ by

$$D(x) := \lim_{t\to 0} \frac{A_t(x) - Id}{t}. \tag{2.2}$$

**Theorem 2.7** *Let $X_t$ be a flow defined on a positive invariant subset $U$, $A_t(x)$ a cocycle over $X_t$ on $U$ and $D(x)$ its infinitesimal generator. Then*

(1) *$\partial_t \mathcal{J}(A_t(x)v) = \langle \tilde{J}_{X_t(x)} A_t(x)v, A_t(x)v\rangle$ for all $v \in E_x$ and $x \in U$, where*

$$\tilde{J}_x := J \cdot D(x) + D(x)^* \cdot J \tag{2.3}$$

*and $D(x)^*$ denotes the adjoint of the linear map $D(x)\colon E_x \to E_x$ with respect to the adapted inner product at $x$;*

(2) *the cocycle $A_t(x)$ is $\mathcal{J}$-separated if, and only if, there exists a neighborhood $V$ of $\Lambda$, $V \subset U$ and a function $\delta\colon V \to \mathbb{R}$ such that*

$$\tilde{J}_x \ge \delta(x)J \quad \textit{for all} \quad x \in V. \tag{2.4}$$

*In particular we get $\partial_t \log|\mathcal{J}(A_t(x)v)| \ge \delta(X_t(x))$, $v \in E_x, x \in V, t \ge 0$;*

(3) *if the inequalities in the previous item are strict, then the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated. Reciprocally, if $A_t(x)$ is strictly $\mathcal{J}$-separated, then there exists compatible field of forms $\mathcal{J}_0$ on $V$ satisfying the strict inequalities of item (2).*

(4) *Define the function*

$$\Delta_s^t(x) := \int_s^t \delta(X_s(x))\, ds. \tag{2.5}$$

*For a $\mathcal{J}$-separated cocycle $A_t(x)$, we have $\frac{|\mathcal{J}(A_{t_2}(x)v)|}{|\mathcal{J}(A_{t_1}(x)v)|} \geq \exp \Delta_{t_1}^{t_2}(x)$ for all $v \in E_x$ and reals $t_1 < t_2$ so that $\mathcal{J}(A_t(x)v) \neq 0$ for all $t_1 \leq t \leq t_2$.*

(5) *if $A_t(x)$ is $\mathcal{J}$-separated and $x \in \Lambda(U)$, $v \in C_+(x)$ and $w \in C_-(x)$ are non-zero vectors, then for every $t > 0$ such that $A_s(x)w \in C_-(X_s(x))$ for all $0 < s < t$*

$$\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)} \leq \frac{|\mathcal{J}(w)|}{\mathcal{J}(v)} \exp\bigl(2\Delta_0^t(x)\bigr). \tag{2.6}$$

(6) *we can bound $\delta$ at every $x \in \Gamma$ by $\sup_{v \in C_-(x)} \frac{\mathcal{J}'(v)}{\mathcal{J}(v)} \leq \delta(x) \leq \inf_{v \in C_+(x)} \frac{\mathcal{J}'(v)}{\mathcal{J}(v)}$.*

*Remark 2.8* If $\delta(x) = 0$, then $\tilde{\mathcal{J}}_x$ is positive semidefinite operator. But for $\delta(x) \neq 0$ the symmetric operator $\tilde{\mathcal{J}}_x$ might be an indefinite quadratic form.

*Remark 2.9* The necessary condition in item (3) of Theorem 2.7 is proved in Sect. 2.5 after Theorem 2.17 and Proposition 2.21.

*Remark 2.10* We can take $\delta(x)$ as a continuous function of the point $x \in U$ by the last item of Theorem 2.7.

*Remark 2.11* Complementing Remark 1.2, the necessary and sufficient condition in items (2–3) of Theorem 2.7, for (strict) $\mathcal{J}$-separation, shows that a cocycle $A_t(x)$ is (strictly) $\mathcal{J}$-separated if, and only if, its inverse $A_{-t}(x)$ is (strictly) $(-\mathcal{J})$-separated.

*Remark 2.12* The inequality (2.10) shows that $\delta$ is a measure of the "minimal instantaneous expansion rate" of $|\mathcal{J} \circ A_t(x)|$ on positive vectors; item (6) of Theorem 2.7 shows that $\delta$ is also a "maximal instantaneous expansion rate" of $|\mathcal{J} \circ A_t(x)|$ on negative vectors; and the last inequality shows in addition that $\delta$ is also a bound for the "instantaneous variation of the ratio" between $|\mathcal{J} \circ A_t(x)|$ on negative and positive vectors.

Hence, the behavior of the area under the function $\delta$, given by $\Delta(x,t) = \int_0^t \delta(X_s(x))\, ds$ as $t$ tends to $\pm\infty$, defines the type of partial hyperbolic splitting (with contracting or expanding subbundles) exhibited by a strictly $\mathcal{J}$-separated cocycle.

In this way we have a condition ensuring partial hyperbolicity of an invariant subset involving only the spatial derivative map of the vector field.

*Proof of Theorem 2.7* The map $\psi(t,v) := \langle JA_t(x)v, A_t(x)v\rangle$ is smooth and for $v \in E_x$ satisfies

$$\begin{aligned}\partial_t \psi(t,v) &= \langle (J \cdot D(X_t(x)))A_t(x)v, A_t(x)v\rangle \\ &\quad + \langle J \cdot A_t(x)v, D(X_t(x))A_t(x)v\rangle \\ &= \langle (J \cdot D(X_t(x)) + D(X_t(x))^* \cdot J)A_t(x)v, A_t(x)v\rangle,\end{aligned}$$

where we have used the fact that the cocycle has an infinitesimal generator $D(x)$: we have the relation

$$\partial_t A_t(x)v = D(X_t(x)) \cdot A_t(x)v \quad \text{for all } t \in \mathbb{R}, x \in M \text{ and } v \in E_x. \tag{2.7}$$

This is because we have the linear variation equation: $A_t(x)$ is the solution of the following non-autonomous linear equation

$$\begin{cases} \dot{Y} = D(X_t(x))Y \\ Y(0) = Id \end{cases} . \tag{2.8}$$

We note that the argument does not change for $x = \sigma$ a singularity of $X_t$.

This proves the first item of the statement of the theorem.

We observe that the independence of $J$ from $X_t(x)$ is a consequence of the choice of adapted coordinates and inner product, since in this setting the operator $J$ is fixed. However, in general, this demands the rewriting of the cocycle in the coordinate system adapted to $\mathcal{J}$.

For the second item, let us assume that $A_t(x)$ is $\mathcal{J}$-separated on $U$. Then, by definition, if we fix $x \in U$

$$\langle JA_t(x)v, A_t(x)v\rangle > 0 \quad \text{for all } t > 0 \quad \text{and} \quad \text{all } v \in E_x \text{ such that } \langle Jv, v\rangle > 0. \tag{2.9}$$

We also note that, by continuity, we have $\langle JA_t(x)v, A_t(x)v\rangle \geq 0$ for all $v$ such that $\langle Jv, v\rangle = 0$. Indeed, for any given $t > 0$ and $v \in C_0$ we can find $w \in C_+$ such that $v + w \in C_+$. Then we have $\langle JA_t(x)(v + \lambda w), A_t(x)(v + \lambda w)\rangle > 0$ for all $\lambda > 0$, which proves the claim letting $\lambda$ tend to 0.

The map $\psi(t, v)$ satisfies $\psi(0, v) = 0 \leq \psi(t, v)$ for all $t > 0$ and $v \in C_0(x)$, hence from the first item already proved

$$0 \leq \partial_t \psi(t, v) \mid_{t=0} = \langle (J \cdot D(x) + D(x)^* \cdot J)v, v\rangle.$$

According to Lemma 2.2 (cf. also Remark 2.3) there exists $\delta(x) \in \mathbb{R}$ such that (2.4) is true and this, in turn, implies that $\partial_t \mathcal{J}(A_t(x)v) \geq \delta(x)\mathcal{J}(A_t(x)v)$, for all $v \in E_x$, $x \in U$, $t \geq 0$. This completes the proof of necessity in the second item.

Moreover, from Lemma 2.2 we have that $\delta(x)$ satisfies the inequalities in item (6).

To see that this is a sufficient condition for $\mathcal{J}$-separation, let $\tilde{\mathcal{J}}_x \geq \delta(x)\mathcal{J}$, for some function $\delta\colon U \to \mathbb{R}$. Then, for all $v \in E_x$ such that $\langle Jv, v\rangle > 0$, since $\partial_t \mathcal{J}(A_t(x)v) \geq \delta(X_t(x))\mathcal{J}(A_t(x)v)$, we obtain

$$|\mathcal{J}(A_t(x)v)| \geq |\mathcal{J}(v)| \exp\left(\int_0^t \delta(X_s(x))\, ds\right) = |\mathcal{J}(v)| \exp \Delta(x, t) > 0, \quad t \geq 0, \tag{2.10}$$

and $\mathcal{J}(A_t(x)v) > 0$ for all $t > 0$ by continuity. This shows that $A_t(x)$ is $\mathcal{J}$-separated. This completes the proof of the second item in the statement of the theorem.

For the third item, we only prove the first statement and leave the longer proof of the second statement for Sect. 2.5 in Proposition 2.21. If $\tilde{J}_x > \delta(x)J$ for all $x \in U$, then for $t > 0$ we obtain (2.10) with strict inequalities for $v \in C_0(x)$, hence $\mathcal{J}(A_t(x)v) > 0$ for $t > 0$. So $A_t(x)$ is strictly $\mathcal{J}$-separated.

For the fourth item, we just itegrate the inequality of item (2) from $s$ to $t$ in the real line.

For the fifth item, we calculate the derivative of the ratio of the forms and use the previous results as follows

$$\partial_t\left(\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)}\right)=-\frac{\langle\tilde{J}_{X_t(x)}A_t(x)w,A_t(x)w\rangle}{\mathcal{J}(A_t(x)v)}-\frac{\mathcal{J}(A_t(x)w)}{\mathcal{J}(A_t(x)v)}\cdot\frac{\langle\tilde{J}_{X_t(x)}A_t(x)v,A_t(x)v\rangle}{\mathcal{J}(A_t(x)v)}$$
$$\le-\delta(X_t(x))\frac{\mathcal{J}(A_t(x)w)}{\mathcal{J}(A_t(x)v)}-\frac{\mathcal{J}(A_t(x)w)}{\mathcal{J}(A_t(x)v)}\cdot\delta(X_t(x))$$
$$=2\delta(X_t(x))\cdot\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)}$$

and the result is obtained by integrating this equation from 0 to $t$. The proof is complete. □

In Sect. 2.6 we show that the asymptotic behavior of the function $\Delta_s^t(x)$ as $t-s$ grows to $\pm\infty$ defines the type of partial hyperbolic splitting exhibited by a strictly $\mathcal{J}$-separated cocycle.

In this way we have a condition ensuring partial and uniform hyperbolicity of an invariant subset involving only the vector field and its spatial derivative map.

## 2.4 Strict $\mathcal{J}$-separated cocycles and domination

We assume from now on that a family $A_t(x)$ of linear multiplicative cocycles on a vector bundle $E_U$ over the flow $X_t$ on a trapping region $U\subset M$ has been given, together with a field of non-degenerate quadratic forms $\mathcal{J}$ on $E_U$ with constant index $q<\dim E_U$.

**Theorem 2.13** *The cocyle $A_t(x)$ is strictly $\mathcal{J}$-separated if, and only if, $E_U$ admits a dominated splitting $F_-\oplus F_+$ with respect to $A_t(x)$ on the maximal invariant subset $\Lambda$ of $U$, with constant dimensions* $\dim F_-=q$, $\dim F_+=p$, $\dim M=p+q$.

Moreover the properties stated in Theorem 2.13 are robust: they hold for all nearby cocycles on $E_U$ over all flows close enough to $X_t$; see Sect. 2.5.

We now start the proof of Theorem 2.13. We construct a decomposition of the tangent space over $\Lambda$ into a direct sum of invariant subspaces and then we prove that this is a dominated splitting.

### *2.4.1 The cones are contracted*

To obtain the invariant subspaces, we show that the action of $A_t(x)$ on the set of all $p$-dimensional spaces inside the positive cones is a contraction in the appropriate distance. For that we use a result from [52].

Let us fix $C_+=C_+(x)$ for some $x\in\Lambda$ and consider the set $G_p(C_+)$ of all $p$-subspaces of $C_+$, where $p=n-q$. This manifold can be identified with the set of all $q\times p$ matrices $T$ with real entries such that $T^*T<I_p$, where $I_p$ is the $p\times p$ identity matrix and $<$ indicates that for the standard inner product in $\mathbb{R}^p$ we have $\langle T^*Tu,u\rangle<\langle u,u\rangle$, for all $u\in\mathbb{R}^p$.

A $\mathcal{J}$-separated operator naturally sends $G_p(C_+)$ inside itself. This operation is a contraction.

**Theorem 2.14** *There exists a distance* dist *on $G_p(C_+)$ so that $G_p(C_+)$ becomes a complete metric space and, if $L\colon V\to V$ is $\mathcal{J}$-separated and $T_1,T_2\in G_p(C_+)$, then*

$$\operatorname{dist}(L(T_1),L(T_2))\le\frac{r_-}{r_+}\operatorname{dist}(T_1,T_2),$$

*where $r_\pm$ are given by Proposition 2.4.*

*Proof* See [52, Theorem 1.6]. □

*2.4.2 Invariant directions*

Now we consider a pair $C_-(x)$ and $C_-(X_{-t}(x))$ of positive cones, for some fixed $t > 0$ and $x \in \Lambda$, together with the linear isomorphism $A_{-t}(x)\colon E_x \to E_{X_{-t}(x)}$. We note that the assumption of strict $\mathcal{J}$-separation ensures that $A_{-t}(x) \mid C_-(x)\colon C_-(x) \to C_-(X_{-t}(x))$. We have in fact

$$\overline{A_{-t}(x) \cdot C_-(x)} \subset C_-(X_{-t}(x)). \tag{2.11}$$

Moreover, by Theorem 2.14 we have that the diameter of $A_{-nt}(x) \cdot C_-(X_{nt}(x))$ decreases exponentially fast when $n$ grows. Hence there exists a unique element $F_-(x) \in G_q(C_-(x))$ in the intersection of all these cones. Analogous results hold for the positive cone with respect to the action of $A_t(x)$. It is easy to see that

$$A_t(x) \cdot F_\pm(x) = F_\pm(X_t(x)), \quad x \in \Lambda. \tag{2.12}$$

Moreover, since the strict inclusion (2.11) holds for whatever $t > 0$ we fix, then we see that the subspaces $F_\pm$ do not depend on the chosen $t > 0$.

*2.4.3 Domination*

The contraction property on $C_+$ for $A_t(x)$ and on $C_-$ for $A_{-t}(x)$, any $t > 0$, implies domination directly. Indeed, let us fix $t > 0$ in what follows and consider the norm $|\cdot|$ induced on $E_x$ for each $x \in U$ by

$$|v| := \sqrt{\mathcal{J}(v_-)^2 + \mathcal{J}(v_+)^2} \quad \text{where} \quad v = v_- + v_+, v_\pm \in F_\pm(x).$$

Now, according to Lemma 2.2 together with Proposition 2.4 we have that, for each $x \in \overline{X_t(U)}$ and every pair of unit vectors $u \in F_-(x)$ and $v \in F_+(x)$

$$\frac{|A_t(x)u|}{|A_t(x)v|} \le \frac{r_-^t(x)}{r_+^t(x)} \le \omega_t := \sup_{z \in \overline{X_t(U)}} \frac{r_-^t(z)}{r_+^t(z)} < 1,$$

where $r_\pm^t(x)$ represent the values $r_\pm$ shown to exist by Lemma 2.2 with respect to the strictly $\mathcal{J}$-separated linear map $A_t(x)$. The value of $\omega_t$ is strictly smaller than 1 by continuity of the functions $r_\pm$ on the compact subset $\overline{X_t(U)}$.

Now we use the following well-known lemma.

**Lemma 2.15** *Let $X_t$ be a $C^1$ flow and $\Lambda$ a compact invariant set for $X_t$ admitting a continuous invariant splitting $T_\Lambda M = F_- \oplus F_+$. Then this splitting is dominated if, and only if, there exists a Riemannian metric on $\Lambda$ inducing a norm such that*

$$\lim_{t\to+\infty} \|A_t(x) \mid_{F_-(x)} \| \cdot \|A_{-t}(X_t(x)) \mid_{F_+(X_t(x))} \| = 0,$$

*for all $x \in \Lambda$.*

This shows that for the cocycle $A_t(x)$ the splitting $E_\Lambda = F_- \oplus F_+$ is dominated, since the above argument does not depend on the choice of $t > 0$ and implies that

$$\lim_{n\to+\infty} \frac{|A_{nt}(x)u|}{|A_{nt}(x)v|} = 0,$$

and we conclude that

$$\lim_{t\to+\infty} \frac{|A_t(x)u|}{|A_t(x)v|} = 0, \quad x \in \Lambda, u \in F_-(x), v \in F_+(x).$$

### 2.4.4 Continuity of the splitting

The continuity of the subbundles $F_\pm$ over $\Lambda$ is a consequence of domination together with the observation that the dimensions of $F_\pm(x)$ do not depend on $x \in \Lambda$; see for example [7, Appendix B]. Moreover, since we are assuming that $A_t(x)$ is smooth, i.e. the cocycle admits an infinitesimal generator, then the $A_t(x)$-invariance ensures that the subbundles $F_\pm(x)$ can be differentiated along the orbits of the flow.

This completes the proof that strict $\mathcal{J}$-separation implies a dominated splitting, as stated in Theorem 2.13.

## 2.5 Domination implies strict $\mathcal{J}$-separation

Now we start the proof of the converse of Theorem 2.13 by showing that, given a dominated decomposition of a vector bundle over a compact invariant subset $\Lambda$ of the base, for a $C^1$ vector field $X$ on a trapping region $U$, there exists a smooth field of quadratic forms $\mathcal{J}$ for which $Y$ is strictly $\mathcal{J}$-separated on $E_V$ over a neighborhood $V$ of $\Lambda$ for each vector field $Y$ sufficiently $C^1$ close to $X$ and every cocycle close enough to $A_t$.

We define a distance between smooth cocycles as follows. If $D_A(x), D_B(x) \colon E_x \to E_x$ are the infinitesimal generators of the cocycles $A_t(x)$, $B_t(x)$ over the flow of $X$ and $Y$ respectively, then we can recover the cocycles through the non-autonomous ordinary differential equation (2.8). We then define the *distance $d$ between the cocycles $A_t$ and $B_t$* to be

$$d\big((A_t)_t, (B_t)_t\big) := \sup_{x\in M} \|D_A(x) - D_B(x)\|,$$

where $\|\cdot\|$ is a norm on the vector bundle $E$. We always assume that we are given a Riemannian inner product in $E$ which induces the norm $\|\cdot\|$.

As before, let $\Lambda = \Lambda(U)$ be a maximal positively invariant subset for a $C^1$ vector field $X$ endowed with a linear multiplicative cocycle $A_t(x)$ defined on a vector bundle over $U$.It is well-known that attracting sets are persistent in the following sense. Let $U$ be a trapping region for the flow of $X$ and $\Lambda(U) = \Lambda_X(U)$ the corresponding attracting set.

**Lemma 2.16** [36, Chapter 10] *There exists a neighborhood $\mathcal{U}$ of $X$ in $\mathfrak{X}^1(M)$ and an open neighborhood $V$ of $\Lambda(U)$ such that $V$ is a trapping region for all $Y \in \mathcal{U}$, that is, there exists $t_0 > 0$ for which*

- $Y_t(V) \subset V \subset U$ *for all* $t > 0$;
- $\overline{Y_t(V)} \subset V$ *for all* $t > t_0$; *and*
- $\overline{Y_t(V)} \subset U$ *for all* $t > 0$.

We can thus consider $\Lambda_Y = \Lambda_Y(U) = \cap_{t>0}\overline{Y_t(U)}$ in what follows for $Y \in \mathcal{V}$ in a small enough $C^1$ neighborhood of $X$.

**Theorem 2.17** *Suppose that $\Lambda$ has a dominated splitting $E_\Lambda = F_- \oplus F_+$. Then there exists a $C^1$ field of quadratic forms $\mathcal{J}$ on a neighborhood $V \subset U$ of $\Lambda$, a $C^1$-neighborhood $\mathcal{V}$ of $X$ and a $C^0$-neighborhood $\mathcal{W}$ of $A_t(x)$ such that $B_t(x)$ is strictly $\mathcal{J}$-separated on $V$ with respect to $Y \in \mathcal{V}$ and $B \in \mathcal{W}$. More precisely, there are constants $\kappa, \omega > 0$ such that, for each $Y \in \mathcal{V}$, $B \in \mathcal{W}$, $x \in \Lambda_Y$ and $t \geq 0$*

$$|\mathcal{J}(B_t(x)v_-)| \leq \kappa e^{-\omega t}\mathcal{J}(B_t(x)v_+),\ v_\pm \in F^B_\pm(x), \quad \mathcal{J}(v_\pm) = \pm 1;$$

*where $F^B_\pm$ are the subbundles of the dominated splitting of $E_{\Lambda_Y}$.*

The quadratic form $\mathcal{J}$ is a inner product in each $F_\pm$, since $F_\pm$ are finite dimensional subbundles of $E$ where $\mathcal{J}$ does not change sign, thus the compactness of $\Lambda$ ensures the following.

**Lemma 2.18** *There exists a constant $K > 0$ such that for every pair of non-zero vectors $(w, v) \in F_-(x) \times F_+(x)$ we have $\frac{1}{K}\|w\|^2 \le |\mathcal{J}(w)| \le K\|w\|^2$, $\frac{1}{K}\|v\|^2 \le \mathcal{J}(v) \le K\|v\|^2$ and*

$$\frac{1}{K}\sqrt{\frac{|\mathcal{J}(w)|}{\mathcal{J}(v)}} \le \frac{\|w\|}{\|v\|} \le K\sqrt{\frac{|\mathcal{J}(w)|}{\mathcal{J}(v)}}.$$

To prove Theorem 2.17 we use the following result from [14], ensuring the existence of adapted metrics for dominated splittings over Banach bundle automorphisms and flows.

Let $\Lambda$ be a compact invariant set for a $C^1$ vector field $X$ and let $E$ be a vector bundle over $M$.

**Theorem 2.19** *Suppose that $T_\Lambda M = F_- \oplus F_+$ is a dominated splitting for a linear multiplicative cocycle $A_t(x)$ over $E$. There exists a neighborhood $V$ of $\Lambda$ and a Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ inducing a norm $|\cdot|$ on $E_V$ such that there exists $\lambda > 0$ satisfying for all $t > 0$ and $x \in \Lambda$*

$$|A_t(x)|_{F_-(x)}| \cdot \left|(A_t(x)|_{F_+(x)})^{-1}\right| < e^{-\lambda t}.$$

*Remark 2.20* A similar result holds for the existence of adapted metric for partially hyperbolic and for uniformly hyperbolic splittings. Moreover, in the adapted metric the bundles $F_\pm$ over $\Lambda$ are almost orthogonal, that is, given $\epsilon > 0$ it is possible to construct such metrics so that $|\langle\langle v_-, v_+\rangle\rangle| < \epsilon$ for all $v_\pm \in F_\pm$ with $\mathcal{J}(v_\pm) = \pm 1$. However this property will not be used in what follows.

We may assume, without loss of generality, that $V$ given by Theorem 2.19 coincides with $U$. Now, we use the adapted Riemannian metric to define the quadratic form on a smaller neighborhood of $\Lambda$ inside $U$.

### *2.5.1 Construction of the field of quadratic forms*

First we choose a continuous field of orthonormal basis (with respect to the adapted metric) $\{e_1(x), \dots, e_s(x)\}$ of $F_-(x)$ and $\{e_{s+1}(x), \dots, e_{s+c}(x)\}$ of $F_+(x)$ for $x \in \Lambda$, where $s = \dim F_-$ and $c = \dim F_+$. Then $\{e_i(x)\}_{i=1}^{s+c}$ is a basis for $E_x$, $x \in \Lambda$.

Secondly, we consider the following quadratic forms

$$\mathcal{J}_x(v) = \mathcal{J}_x\left(\sum_{i=1}^{s+c}\alpha_i e_i(x)\right) := |v^+|^2 - |v^-|^2 = \sum_{i=s+1}^{s+c}\alpha_i^2 - \sum_{i=1}^{s}\alpha_j^2, \quad v \in E_x, x \in V,$$

where $v^\pm \in F_\pm(x)$ are the unique orthogonal projections on the subbundles such that $v = v^- + v^+$. This defines a field of quadratic forms on $\Lambda$.

We note that, since $F_- \oplus F_+$ is $A_t(x)$-invariant over $\Lambda$, and the vector field $X$ and the flow $X_t$ are $C^1$, the field of quadratic forms constructed above is differentiable along the flow direction, because $F_\pm(X_t(x)) = A_t(x) \cdot F_\pm(x)$ is differentiable in $t \in \mathbb{R}$ for each $x \in \Lambda$.

Clearly $F_-$ is a $\mathcal{J}$-negative subspace and $F_+$ is a $\mathcal{J}$-positive subspace, which shows that the index of $\mathcal{J}$ equals $s$ and that the forms are non-degenerate.

In addition, we have strict $\mathcal{J}$-separation over $\Lambda$. Indeed, $v = v^- + v^+ \in C_+(x) \cup C_0(x)$ for $x \in \Lambda$ means $|v^+| \geq |v^-|$ and the $A_t(x)$-invariance of $F_\pm$ ensures that $A_t(x)v = A_t(x)v^- + A_t(x)v^+$ with $A_t(x)v^\pm \in F_\pm(X_t(x))$ and $\sqrt{\mathcal{J}(A_t(x)v^+)} = |A_t(x)v^+| > e^{\lambda t}|A_t(x)v^-| = \sqrt{|\mathcal{J}(A_t(x)v^-)|}$, so that $A_t(x)v \in C_+(X_t(x))$.

We are ready to obtain the reciprocal of item 3 of Theorem 2.7.

**Proposition 2.21** *If the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated over a compact $X_t$-invariant subset $\Lambda$, then there exist a compatible field of quadratic forms $\mathcal{J}_0$ and a function $\delta\colon \Lambda \to \mathbb{R}$ such that $\tilde{\mathcal{J}}_{0,x} > \delta(x)\mathcal{J}_0$ for all $x \in \Lambda$.*

*Proof* We have already shown that a strictly $\mathcal{J}$-separated cocycle has a dominated splitting $E = F_- \oplus F_+$ in Sect. 2.4. Then we build the field of quadratic forms $\mathcal{J}_0$ according to the previous arguments in this section, and calculate for $v_0 \in C_0(x)$, $v_0 = v^- + v^+$ with $v^\pm \in F_\pm(x)$ and $|v^\pm| = 1$, for a given $x \in \Lambda$ and all $t > 0$

$$\mathcal{J}_0(A_t(x)v_0) = |A_t(x)v^-|^2 \left( \frac{|A_t(x)v^+|^2}{|A_t(x)v^-|^2} - 1 \right) \geq |A_t(x)v^-|^2 \cdot (e^{2\lambda t} - 1). \tag{2.13}$$

The derivative of the right hand side above satisfies

$$2\lambda e^{2\lambda t}|A_t(x)v^-|^2 + (e^{2\lambda t} - 1)\partial_t|A_t(x)v^-|^2 \xrightarrow[t\searrow 0]{} 2\lambda.$$

Since the left hand side and the right hand side of (2.14) have the same value at $t = 0$ (we note that $\mathcal{J}_0(v_0) = 0$ by the choice of $v_0$), we have

$$\tilde{\mathcal{J}}_x(v_0) = \partial_t \mathcal{J}_0(A_t(x)v_0)\mid_{t=0} \geq 2\lambda > 0, \quad x \in \Lambda.$$

Thus, $\tilde{\mathcal{J}}_x(v_0) > 0$ for $\vec{0} \neq v_0 \in C_0(x)$ which implies by Lemma 2.2 that $\tilde{\mathcal{J}}_x > \delta(x)\mathcal{J}_0$ for some real function $\delta(x)$. Finally, the quadratic forms $\mathcal{J}_0$ and $\mathcal{J}$ are compatible. □

### *2.5.2 Continuous/smooth extension to a neighborhood*

We recall that the adapted Riemannian metric is defined on a neighborhood $V$ of $\Lambda$. We can write $\mathcal{J}_x(v) = \langle\langle J_x(v), v\rangle\rangle_x$ for all $v \in T_xM$, $x \in \Lambda$, where $J_x\colon T_xM \circlearrowleft$ is a self-adjoint operator. This operator can be represented by a matrix (with respect to the basis adapted to $\mathcal{J}_x$) whose entries are continuous functions of $x \in \Lambda$.

These functions can be extended to continuous functions on $V$ yielding a continuous extension $\hat{\mathcal{J}}$ of $\mathcal{J}$. We recall that the field $\mathcal{J}$ is differentiable along the flow direction. Thus $\hat{\mathcal{J}}$ remains differentiable along the flow direction over the points of $\Lambda$.

Finally, these functions can then be $C^1$ regularized so that they become $\epsilon$-$C^0$-approximated by $C^1$ functions on $V$. We obtain in this way a smooth extension $\bar{\mathcal{J}}$ of $\mathcal{J}$ to a neighborhood of $\Lambda$ in such a way that $\bar{\mathcal{J}}$ is automatically $C^1$ close to $\mathcal{J}$ over orbits of the flow on $\Lambda$. This means that, given $\epsilon > 0$, we can find $\bar{\mathcal{J}}$ such that

- $|\hat{\mathcal{J}}_y(v) - \bar{\mathcal{J}}_y(v)| < \epsilon$ for all $v \in E_y$, $y \in V$ ($C^0$-closeness on $V$);
- $|\partial_t \hat{\mathcal{J}}_{X_t(x)}(A_t(x)v) - \partial_t \bar{\mathcal{J}}_{X_t(x)}(A_t(x)v)| < \epsilon$ for all $v \in E_x$, $x \in \Lambda$ and $t \in \mathbb{R}$.

*Remark 2.22* We note that $F_\pm(x)$ are subspaces with the same sign for both $\mathcal{J}$ and $\bar{\mathcal{J}}$. Hence $\pm\mathcal{J}$ and $\pm\bar{\mathcal{J}}$ define inner products in these finite dimensional vector spaces, thus we can find $C_\pm(x)$ such that $C_\pm(x)^{-1}|\mathcal{J}\mid_{F_\pm(x)}| \leq |\bar{\mathcal{J}}\mid_{F_\pm(x)}| \leq C_\pm(x)|\mathcal{J}\mid_{F_\pm(x)}|$. This ensures that $\mathcal{J}$ and $\bar{\mathcal{J}}$ are equivalent forms over $\Lambda$: since $\mathcal{J}$ and $\bar{\mathcal{J}}$ are continuous on $\Lambda$ we just have to take $C = \max\{C_\pm(x) \colon x \in \Lambda\}$. Moreover we also have that the Riemannian norm $\|\cdot\|$ of $M$ and the adapted norm $|\cdot|$ are also equivalent: we can assume that $C^{-1}|\cdot| \leq \|\cdot\| \leq C|\cdot|$.

Using the compatibility between $\mathcal{J}$ and $\bar{\mathcal{J}}$ we obtain for $v_\pm \in F_\pm(x)$

$$\begin{aligned}|\bar{\mathcal{J}}(A_t(x)v_-)| &\leq C|\mathcal{J}(A_t(x)v_-)| \leq C\cdot K\|A_t(x)v_-\|^2 \leq CK\cdot C^2|A_t(x)v_-|^2\\ &\leq C^3K\cdot e^{-2\lambda t}|A_t(x)v_+|^2 \leq C^4Ke^{-2\lambda t}\|A_t(x)v_+\|^2\\ &\leq C^4Ke^{-2\lambda t}\cdot K\cdot\mathcal{J}(A_t(x)v_+) \leq C^4K^2e^{-2\lambda t}\cdot C\bar{\mathcal{J}}(A_t(x)v_+)\\ &\leq \kappa e^{-2\lambda t}\bar{\mathcal{J}}(A_t(x)v_+)\end{aligned}$$

for all $t > 0$, where we used Lemma 2.18 together with Theorem 2.19.

Therefore we have the relations in the statement of Theorem 2.17 if we set $\omega = 2\lambda$.

#### 2.5.3 *Strict separation for the extension/smooth approximation*

We now show that $Y$ is strictly $\bar{\mathcal{J}}$-separated on $V$ for every vector field $Y$ in a neighborhood $\mathcal{V}$ of $X$ and for every multiplicative cocycle $B_t(x)$ over $Y$ which is $C^0$ close to $A_t(x)$.

We start by observing that $\bar{\mathcal{J}}$ is differentiable along the flow direction. Then we note that, from Proposition 2.21

$$\iota := \inf\{\widetilde{\mathcal{J}}_x - \delta(x)\mathcal{J}_x : x \in \Lambda\} > 0$$

and recall that $\widetilde{J}_x = J_x\cdot D(x) + D(x)^*\cdot J_x$. Hence, by choosing $V$ sufficiently small around $\Lambda$, we obtain

$$\hat{\iota} = \inf\{\widetilde{\hat{\mathcal{J}}}_y - \delta(y)\hat{\mathcal{J}}_y : y \in V\} \geq \frac{\iota}{2} > 0,$$

since $\hat{\mathcal{J}}$ is an extension of $\mathcal{J}$ on $\Lambda$, and the function $\delta$ is defined by $\hat{J}$ and $D(x)$ according to Remark 2.10. Finally, by taking a sufficiently small $\epsilon > 0$ in the choice of the $C^1$ approximation $\bar{J}$, we also get

$$\bar{\iota} = \inf\{\widetilde{\bar{\mathcal{J}}}_y - \delta(y)\bar{\mathcal{J}}_y : y \in V\} \geq \frac{\hat{\iota}}{2} \geq \frac{\iota}{4} > 0.$$

From Theorem 2.7 and Proposition 2.21, we know that this is a necessary and sufficient condition for strict $\bar{\mathcal{J}}$-separation of $A_t(x)$ over $V$.

#### 2.5.4 *Strict separation for nearby flows/cocycles*

Given a vector field $X$ on $M$ and a linear multiplicative cocycle $A_t(x)$ on a vector bundle $E$ over $M$, for a $C^1$ close vector field $Y$ and a $C^0$ close cocycle $B_t(x)$ over $Y$, the infinitesimal generator $D_{B,Y}(x)$ of $B_t(x)$ will be a linear map close to the infinitesimal generator $D(x)$ of $A_t(x)$ at $x$. That is, given $\epsilon > 0$ we can find a $C^1$ neighborhood $\mathcal{V}$ of $X$ and a $C^0$ neighborhood $\mathcal{W}$ of the cocycle $A$ such that

$$(Y, B) \in \mathcal{V}\times\mathcal{W} \implies \|D_{B,Y}(x) - D(x)\| < \epsilon, \quad x \in M.$$

Hence, since $\delta$ also depends continuously on the infinitesimal generator, we obtain

$$\tilde{\iota} = \inf\{\widetilde{\bar{\mathcal{J}}}_y - \delta_{B,Y}(y)\bar{\mathcal{J}}_y : y \in V, Y \in \mathcal{V}, B \in \mathcal{W}\} \geq \frac{\bar{\iota}}{2} > 0.$$

This shows that we have strict $\bar{\mathcal{J}}$ separation for all nearby cocycles over all $C^1$-close enough vector fields over the same neighborhood $V$ of the original invariant attracting set $\Lambda$.

Finally, to obtain the inequalities of the statement of Theorem 2.17, since we have strict $\bar{\mathcal{J}}$-separation for $B_t(x)$, we also have a dominated splitting $E_{\Lambda_Y} = F_-^B \oplus F_+^B$ over $\Lambda_Y$ whose subbundles have the same sign as the original $F_\pm$ subbundles of $E_\Lambda$ for $A_t(x)$. We can then repeat the arguments leading to the constant $\kappa$, which depends continuously on $Y$.

This completes the proof of Theorem 2.17.

## 2.6 Characterization of the splitting through the function $\delta$

We now use the area under the function $\delta$ to characterize different dominated splittings that may arise in our setting.

**Theorem 2.23** *Let $\Gamma$ be a compact invariant set for $X_t$ admitting a dominated splitting $E_\Gamma = F_- \oplus F_+$ for $A_t(x)$, a linear multiplicative cocycle over $\Gamma$ with values in $E$. Let $\mathcal{J}$ be a $C^1$ field of indefinite quadratic forms such that $A_t(x)$ is strictly $\mathcal{J}$-separated. Then*

(1) *$F_- \oplus F_+$ is partially hyperbolic with $F_-$ not uniformly contracting and $F_+$ uniformly expanding if, and only if, $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} +\infty$ for all $x \in \Gamma$.*
(2) *$F_- \oplus F_+$ is partially hyperbolic with $F_-$ uniformly contracting and $F_+$ not uniformly expanding if, and only if, $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} -\infty$ for all $x \in \Gamma$.*
(3) *$F_- \oplus F_+$ is hyperbolic (that is, $F_-$ is uniformly contracted and $F_+$ is uniformly expanded) if, and only if, there exists a compatible field of quadratic forms $\mathcal{J}_0$ in a neighborhood of $\Gamma$ such that $\mathcal{J}_0'(v) > 0$ for all $v \in E_x$ and all $x \in \Gamma$.*

Above we write $\mathcal{J}'(v) = < \tilde{J}_x v, v >$ where $\tilde{J}_x$ is given in Proposition 1.3.

In the proof we use the following useful equivalence.

**Lemma 2.24** *Let $F \subset E$ be a continuous $A_t(x)$-invariant subbundle of the finite dimensional vector bundle $E$ with compact base $\Lambda$. Then, there are constants $K, \omega > 0$ satisfying for $\vec{0} \neq v \in F_x, x \in \Lambda, t > 0$*

$$\|A_t(x)v\| \le K e^{-\omega t}\|v\| \quad (\|A_{-t}(x)v\| \le K e^{-\omega t}\|v\|, \ \textit{respectively})$$

*if, and only if, for every $x \in \Lambda$ and $\vec{0} \neq v \in E_x$*

$$\lim_{t\to+\infty} \|A_t(x)v\| = 0 \quad (\lim_{t\to+\infty} \|A_{-t}(x)v\| = 0, \ \textit{respectively}).$$

*Proof* See e.g. [24]. □

*Proof of Theorem 2.23* We consider a compact $X_t$-invariant subset $\Gamma$, a vector bundle $E$ over $\Gamma$ and a linear multiplicative cocycle $A_t(x)$ over $X_t$ with values in $E$.

We fix a $C^1$ field of indefinite quadratic forms $\mathcal{J}$ such that $A_t(x)$ is strictly $\mathcal{J}$-separated and $\mathcal{J}$ is compatible with the splitting $E_\Gamma = F_- \oplus F_+$.

(1) If $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} +\infty$ for all $x \in \Gamma$ then, from Proposition 2.7(4a) we get $\lim_{t\to+\infty} \mathcal{J}(A_t(x)v_+) = +\infty$ for every $\vec{0} \neq v_+ \in F_+(x)$; and this ensures that $F_+$ is uniformly expanded, after Lemma 2.24.
(2) If $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} -\infty$ for all $x \in \Gamma$ then, from Proposition 2.7(4a) we get $\frac{|\mathcal{J}(v_-)|}{|\mathcal{J}(A_t(x)v_-)|} \ge \exp \Delta_t^0(x) = \exp(-\Delta_0^t(x)) \xrightarrow[t\to+\infty]{} +\infty$ and so $|A_t(x)v_-| \xrightarrow[t\to+\infty]{} 0$ for every $\vec{0} \neq v_- \in F_-(x), x \in \Gamma$; and this ensures that $F_-$ is uniformly contracted after Lemma 2.24.

Clearly cases (1) and (2) are mutually exclusive and so each case proves the sufficient conditions of items (1) and (2) of the statement of Theorem 2.23. Reciprocally:

(1) If $F_+$ is formed by uniformly expanded vectors then, for $\vec{0} \neq v_+ \in F_+(x)$ and $s \in \mathbb{R}$

$$\exp \Delta_t^s(x) \leq \frac{|\mathcal{J}(A_s(x)v_+)|}{|\mathcal{J}(A_t(x)v_+)|} \xrightarrow[t\to+\infty]{} 0 \quad \text{thus} \quad \Delta_t^s(x) \xrightarrow[t\to+\infty]{} -\infty.$$

This implies that $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} +\infty$ for all $x \in \Gamma$.

(2) If $F_-$ is formed by uniformly contracted vectors then, for $\vec{0} \neq v_- \in F_-(x)$ and $s \in \mathbb{R}$

$$\exp \Delta_t^s(x) \leq \frac{|\mathcal{J}(A_s(x)v_+)|}{|\mathcal{J}(A_t(x)v_+)|} \xrightarrow[t\to-\infty]{} 0 \quad \text{thus} \quad \Delta_t^s(x) \xrightarrow[t\to-\infty]{} -\infty.$$

This implies that $\Delta_s^t(x) \xrightarrow[(t-s)\to+\infty]{} -\infty$ for all $x \in \Gamma$.

This proves items (1) and (2) of Theorem 2.23.

Now let us assume that $F_- \oplus F_+$ is a uniformly hyperbolic splitting and take $\mathcal{J}_0$ the field of quadratic forms provided by Theorem 2.19, which is compatible with $\mathcal{J}$.

For $v = v_- + v_+ \in E_x$ with $v_\pm \in F_\pm(x)$ and $\mathcal{J}(v) > 0$ (note that the difference below is positive for small $|t|$)

$$\mathcal{J}_0(A_t(x)v) = |A_t(x)v_+|^2 \left(1 - \frac{|A_t(x)v_-|^2}{|A_t(x)v_+|^2}\right) \geq |A_t(x)v_+|^2 \left(1 - e^{-2\lambda t}\frac{|v_-|^2}{|v_+|^2}\right). \tag{2.14}$$

The derivative of the right hand side above satisfies

$$2\lambda e^{2\lambda t}|A_t(x)v_+|^2 \frac{|v_-|^2}{|v_+|^2} + \left(|v_+|^2 - e^{2\lambda t}|v_-|^2\right) \frac{\partial_t |A_t(x)v_+|^2}{|v_+|^2} \xrightarrow[t\searrow 0]{} 2\lambda|v_-|^2$$
$$+\mathcal{J}_0(v)\frac{\partial_t|A_t(x)v_+|^2}{|v_+|^2}.$$

Since the left hand side and the right hand side of (2.14) have the same value at $t = 0$, the limit above is a lower bound for $\partial_t \mathcal{J}_0(A_t(x)v)\mid_{t=0}$. Because $\mathcal{J}_0(A_t(x)v_+) \geq e^{2\lambda t}\mathcal{J}_0(v)$ and $\mathcal{J}_0(v) > 0$

$$\mathcal{J}_0'(v) = \partial_t \mathcal{J}_0(A_t(x)v)\mid_{t=0} \geq 2\lambda|v_-|^2 + 2\lambda\mathcal{J}_0(v) = 2\lambda|v_+|^2 > 0.$$

Moreover, since for any non-zero vector $v_0 = v_- + v_+$ in $C_0(x)$ we can make an arbitrarily small perturbation to $v_-$, keeping $v_+$, so that $\mathcal{J}_0(\tilde{v}_- + v_+) > 0$, we obtain $\mathcal{J}_0'(\tilde{v}_- + v_+) \geq 2\lambda|v_+|^2$ and so $\mathcal{J}_0'(v_0) > 0$ for all non-zero $v_0$ in $C_0(x)$. Now for $\mathcal{J}_0(v) < 0$ we have

$$\begin{aligned}\mathcal{J}_0(A_t(x)v) &= |A_t(x)v_-|^2 \left(\frac{|A_t(x)v_+|^2}{|A_t(x)v_-|^2} - 1\right) \geq |A_t(x)v_-|^2 \left(e^{2\lambda t}\frac{|v_+|^2}{|v_-|^2} - 1\right) \\ &\geq e^{-2\lambda t}|v_-|^2 \left(e^{2\lambda t}\frac{|v_+|^2}{|v_-|^2} - 1\right) = |v_+|^2 - e^{-2\lambda t}|v_-|^2\end{aligned} \tag{2.15}$$

where we used domination in the first inequality and $\mathcal{J}_0(v) < 0$ and $|t|$ small in the second inequality. Hence $\mathcal{J}_0'(v) \geq 2\lambda|v_-|^2 > 0$. This completes the proof of the sufficient condition of item (3).

Reciprocally, let us assume that $\mathcal{J}'$ is a positive definite quadratic form. Hence $\mathcal{J}'$ is an inner product on a finite dimensional vector bundle $E_\Gamma$ with compact base, and so there exists $\kappa > 0$ such that $\mathcal{J}' \geq \kappa|\cdot|^2$ and $\kappa|\mathcal{J}| \leq |\cdot|^2$. Thus $\mathcal{J}'(v) \geq \kappa^2|\mathcal{J}(v)|$ for all $v \in E$, which implies $\mathcal{J}(A_t(x)v_+) \geq e^{\kappa^2 t}\mathcal{J}(v_+)$ for $\vec{0} \neq v_+ \in F_+(x)$; and $|\mathcal{J}(A_t(x)v_-)| \leq e^{-\kappa^2 t}|\mathcal{J}(v_-)|$

for all $\vec{0} \neq v_- \in F_-(x)$. This shows, from the comparison results given in Lemma 2.18, that $F_- \oplus F_+$ is a uniformly hyperbolic splitting of $E_\Gamma$, and completes the proof of Theorem 2.23. □

## 3 Partial hyperbolicity: proof of Theorem A

Now we prove Theorem A. We show that strict $\mathcal{J}$-separation of a $\mathcal{J}$-non-negative flow $X_t$ on a trapping region $U$ implies the existence of a dominated splitting and that the dominating bundle (the one with the weakest contraction) is necessarily uniformly contracting. That is, we have in fact a partially hyperbolic splitting.

The strategy is to consider the derivative cocycle $DX_t$ of the smooth flow $X_t$ in the place of $A_t(x)$ and use the results of Sect. 2.2. In this setting we have that the infinitesimal generator is given by $D(x) = DX(x)$ the spatial derivative of the vector field $X$. Since the direction of the flow $E^X_x := \mathbb{R} \cdot X(x) = \{s \cdot X(x) \colon s \in \mathbb{R}\}$ is $DX_t$-invariant for all $t \in \mathbb{R}$, if $U$ is a trapping region where $X_t$ is $\mathcal{J}$-separated and $\mathcal{J}(X(x)) \geq 0$ for some $x \in U$, then $\mathcal{J}(DX_t(X(x))) \geq 0$ for all $t > 0$ and this function is bounded.

We recall Lemma 2.16 and deduce the following.

**Corollary 3.1** *Let $X_t$ be strictly $\mathcal{J}$-separated on $U$. Then there exist a neighborhood $\mathcal{U}$ of $X$ in $\mathfrak{X}^1(M)$ and a neighborhood $V$ of $\Lambda(U)$ such that $V$ is a trapping region for every $Y \in \mathcal{U}$ and each $Y \in \mathcal{U}$ is strictly $\mathcal{J}$-separated in $V$.*

*Proof* The assumption implies that $\tilde{\mathcal{J}}_x = \tilde{\mathcal{J}}^X_x > \alpha(x)\mathcal{J}$ for all $x \in U$. Let $\mathcal{U}$ and $V$ be the neighborhoods of $X$ and $\Lambda$ given by Lemma 2.16. Let also $Y \in \mathcal{U}$ be fixed.

Writing $\tilde{J}^Y_x := J \cdot DY(x) + DY(x)^* \cdot J$, we can make the norm $\|\tilde{\mathcal{J}}_x - \tilde{\mathcal{J}}^Y_x\|$ smaller than

$$\frac{1}{2} \min \left\{ \inf_{v \in C_+(x)} \frac{\langle \tilde{J}_x v, v\rangle}{\langle J v, v\rangle} - \alpha(x) \colon x \in \overline{U_0} \right\}$$

for all $x \in V$ by shrinking $\mathcal{U}$ if needed. This ensures that there exists $\beta \colon \overline{V} \to \mathbb{R}$ such that $\tilde{\mathcal{J}}^Y_x > \beta(x)\mathcal{J}$ for all $x \in \overline{V}$, so $Y$ is strictly $\mathcal{J}$-separated on $V$. □

From Sect. 2.4 we know that there exists a continuous dominated splitting $F^Y_-(x) \oplus F^Y_+(x)$ of $T_x M$ for $x \in \Lambda_Y(U)$, with respect to $Y_t$ for all $Y \in \mathcal{U}$.

The strict $\mathcal{J}$-separation on $U$ for $X_t$ implies that any invariant subbundle of $T_x M$ along an orbit of the flow $X_t$ must be contained in $F_\pm(x)$. In particular, the characteristic space corresponding to the flow direction is contained in $F_+(x)$.

**Lemma 3.2** *Let $\Lambda$ be a compact invariant set for a flow $X$ of a $C^1$ vector field $X$ on $M$.*

(1) *Given a continuous splitting $T_\Lambda M = E \oplus F$ such that $E$ is uniformly contracted, then $X(x) \in F_x$ for all $x \in \Lambda$.*
(2) *Assuming that $\Lambda$ is non-trivial and has a continuous* and dominated *splitting $T_\Lambda M = E \oplus F$ such that $X(x) \in F_x$ for all $x \in \Lambda$, then $E$ is a uniformly contracted subbundle.*

*Proof* For the first item, we denote by $\pi(E_x) \colon T_x M \to E_x$ the projection on $E_x$ parallel to $F_x$ at $T_x M$, and likewise $\pi(F_x) \colon T_x M \to F_x$ is the projection on $F_x$ parallel to $E_x$. We note that for $x \in \Lambda$

$$X(x) = \pi(E_x) \cdot X(x) + \pi(F_x) \cdot X(x)$$

and for $t \in \mathbb{R}$, by linearity of $DX_t$ and $DX_t$-invariance of the splitting $E \oplus F$

$$\begin{aligned} DX_t \cdot X(x) &= DX_t \cdot \pi(E_x) \cdot X(x) + DX_t \cdot \pi(F_x) \cdot X(x) \\ &= \pi(E_{X_t(x)}) \cdot DX_t \cdot X(x) + \pi(F_{X_t(x)}) \cdot DX_t \cdot X(x) \end{aligned}$$

Let $z$ be a limit point of the negative orbit of $x$, that is, we assume that there is a strictly increasing sequence $t_n \to +\infty$ such that $\lim_{n\to+\infty} x_n := \lim_{n\to+\infty} X_{-t_n}(x) = z$. Then $z \in \Lambda$ and, *if* $\pi(E_x) \cdot X(x) \neq \vec{0}$ we get

$$\lim_{n\to+\infty} DX_{-t_n} \cdot X(x) = \lim_{n\to+\infty} X(x_n) = X(z) \quad \text{but also}$$

$$\|DX_{-t_n} \cdot \pi(E_x) \cdot X(x)\| \geq ce^{\lambda t_n} \|\pi(E_x) \cdot X(x)\| \xrightarrow[n\to+\infty]{} +\infty. \tag{3.1}$$

This is possible only if the angle between $E_{x_n}$ and $F_{x_n}$ tends to zero when $n \to +\infty$.

Indeed, using the Riemannian metric on $T_y M$, the angle $\alpha(y) = \alpha(E_y, F_y)$ between $E_y$ and $F_y$ is related to the norm of $\pi(E_y)$ as follows: $\|\pi(E_y)\| = 1/\sin(\alpha(y))$. Therefore

$$\begin{aligned} \|DX_{-t_n} \cdot \pi(E_x) \cdot X(x)\| &= \|\pi(E_{x_n}) \cdot DX_{-t_n} \cdot X(x)\| \\ &\leq \frac{1}{\sin(\alpha(x_n))} \cdot \|DX_{-t_n} \cdot X(x)\| \\ &= \frac{1}{\sin(\alpha(x_n))} \cdot \|X(x_n)\| \end{aligned}$$

for all $n \geq 1$. Hence, if the sequence (3.1) is unbounded, then $\lim_{n\to+\infty} \alpha(X_{-t_n}(x)) = 0$.

However, since the splitting $E \oplus F$ is continuous over the compact $\Lambda$, the angle $\alpha(y)$ is a continuous and positive function of $y \in \Lambda$, and thus must have a positive minimum in $\Lambda$. This contradiction shows that $\pi(E_x) \cdot X(x)$ is always the zero vector and so $X(x) \in F_x$ for all $x \in \Lambda$.

This completes the proof of the first item.

For the second item, fix $x \in \Lambda$ with $X(x) \neq \vec{0}$, take $v \in E_x$ and use the definition of $(K, \lambda)$-domination to obtain for each $t > 0$

$$Ke^{-\lambda t} \geq \frac{\|DX_t \cdot v\|}{\|DX_t \cdot X(x)\|} = \frac{\|DX_t \cdot v\|}{\|X(X_t(x))\|} \geq \frac{1}{C}\|DX_t \cdot v\|$$

where $C = \sup\{\|X(y)\colon y \in \Lambda\}$ is a positive number. For $\sigma \in \Lambda$ such that $X(\sigma) = \vec{0}$, we fix $T > 0$ such that $CKe^{-\lambda T} < 1/2$ and, since $\Lambda$ is non-trivial, we can find a sequence $x_n \to \sigma$ of regular points of $\Lambda$. The continuity of the derivative cocycle ensures $1/2 \geq \|DX_T \mid E_\sigma\| = \lim_{n\to+\infty} \|DX_T \mid E_{x_n}\|$ and so $E_\sigma$ is also a uniformly contracted subspace. □

Since $X$ and $\Lambda = \Lambda_X$ are in the conditions of the second item of the previous lemma, we have proved partial hyperbolicity for $T_\Lambda M = F_- \oplus F_+$.

At this point, we have concluded the proof of sufficiency in the statement Theorem A.

The necessary part of the statement of Theorem A is a simple consequence of Theorem 2.17 applied to the cocycle $DX_t$ acting on the vector bundle $T_U M$, the tangent bundle on the trapping region $U$.

This completes the proof of Theorem A.

## 3.1 Uniform hyperbolicity

By using Theorem A, we are able to present the proof of the Corollaries B and C.

*Proof of Corollary B* Let $X \in \mathfrak{X}^1(M)$ and $\Lambda$ be the maximal invariant set of a trapping region $U$ for $X$. Consider $\mathcal{J}, \mathcal{G}$ two differentiable fields of non-degenerated quadratic forms on $U$ with constant indices $s$ and $n-s-1$, respectively, where $n = \dim M$ and $s < n-2$. Since $\Lambda \cap \mathrm{Sing}(X) = \emptyset$, the flow direction $X(x)$ is non-zero for all point $x \in \Lambda$ and generates an invariant line bundle $E^X$ over $\Lambda$.

On the one hand, by Remark 2.11, $-X$ is a non-negative strictly separated with respect to $-\mathcal{G}$ on $\Lambda$. Then Theorem A implies that there is a partially hyperbolic splitting $T_\Lambda M = E^{cs} \oplus E^u$ with the subbundle $E^u$ uniformly expanding, and dimensions $\dim E^{cs} = s+1$ and $\dim E^u = n-s-1$. Thus, by Lemma 3.2, the flow direction $E^X$ is contained in $E^{cs}$.

On the other hand, with analogous arguments, we prove that, for $\mathcal{J}$, $\Lambda$ has a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^{cu}$, with $E^s$ uniformly contracting, and so $E^X \subset E^{cu}$, with dimensions $\dim E^s = s$ and $\dim E^{cu} = n-s$.

Moreover we clearly have $E^s \cap E^{cu} = \{0\}$ and $E^u \cap E^{cs} = \{0\}$. Hence we have the following dominated splittings

$$(E^s \oplus E^X) \oplus E^u = E^s \oplus (E^X \oplus E^u) = T_\Lambda M,$$

with $\dim E^{cs} = \dim(E^s \oplus E^X)$ and $\dim E^{cu} = \dim(E^s \oplus E^X)$. By uniqueness of dominated splittings with the same dimensions we obtain $E^{cu} = E^X \oplus E^u$ and $E^{cs} = E^s \oplus E^X$, and the splitting of $T_\Lambda M$ above is a hyperbolic splitting. □

*Proof of Corollary C* Consider $M$ a closed Riemannian manifold with dimension $n \geq 3$ and $X \in \mathfrak{X}^1(M)$ a incompressible vector field.

Let $\mathcal{J}$ be a field of non-degenerate quadratic forms on $M$, with constant index $\mathrm{ind}(\mathcal{J}) = \dim(M) - 2$, such that $X_t$ is a non-negative $\mathcal{J}$-separated flow.

By Theorem A and the hypothesis on $\mathrm{ind}(\mathcal{J})$, we obtain a partially hyperbolic splitting $TM = E \oplus F$, with $\dim E = \dim(M) - 2$ and $\dim F = 2$ and $E$ uniformly contracted. Hence, as the flow is volume-preserving, the area along the two-dimensional direction $F$ is expanded. Indeed, the angle $\theta(E_x, F_x)$ between the subbundles is uniformly bounded away from zero (by domination of the splitting; see [31]) and so

$$1 = |\det DX_t(x)| = |\det DX_t \mid_{E_x} | \cdot |\det DX_t \mid_{F_x} | \cdot \sin\theta(E_{X_t(x)}, F_{X_t(x)})$$

which for $t < 0$ ensures that

$$|\det DX_t \mid_{F_x} | \leq \sin\theta_0 \cdot |\det DX_t \mid_{E_x} |^{-1} \xrightarrow[t \to -\infty]{} 0.$$

Thus, $M$ is a singular-hyperbolic set for $X$. Moreover, there can be no singularities, since they cannot be in the interior of a singular-hyperbolic set; see Doering [13] and Morales–Pacifico–Pujals [29] in dimension three; Li–Gan–Wen [21] and Vivier [48] for higher dimensions; or [5, Chapter 4]. Hence $M$ is a singular-hyperbolic set for $X_t$ without singularities. Therefore $X$ is an Anosov flow. □

## 4 Sectional hyperbolicity: proof of Theorem D

Here we prove Theorem D. We assume that $X$ is a $C^1$ vector field in an open trapping region $U$ with a smooth field of non-degenerate quadratic forms $\mathcal{J}$ such that $X$ is non-negative and strictly $\mathcal{J}$-separated, and the linear Poincaré flow on any compact invariant subset $\Gamma$ of $\Lambda^*_X(U) := \Lambda_X(U) \backslash \mathrm{Sing}(X)$ is strictly $\mathcal{J}_0$-monotone, for some field of quadratic forms $\mathcal{J}_0$ equivalent to $\mathcal{J}$.

We show that, in this setting, the linear Poincaré flow of $X$ on $\Gamma$ has a hyperbolic splitting. After that, as a consequence, we show that either there are no singularities in $\Lambda$ and then $\Lambda$ is a hyperbolic attracting set; or, otherwise, $\Lambda$ is a sectional hyperbolic attracting set, as long as the singularities are sectional hyperbolic with index compatible with the index of the attracting set.

### 4.1 Strict $\mathfrak{J}$-monotonicity for the linear Poincaré flow and hyperbolicity

Strict $\mathfrak{J}$-monotonicity is clearly stronger than strict $\mathfrak{J}$-separation, so that on a compact invariant subset $\Gamma$ of $\Lambda_X^*(U)$ the linear Poincaré flow $P^t$ admits a dominated splitting $N^s \oplus N^u$ of $N$ over $\Gamma$. But with strict $\mathfrak{J}$-monotonicity we can say more.

Consider $X \in \mathfrak{X}^1(M)$ with a trapping region $U$, $\Lambda_X(U)$ its attracting set and a smooth field of non-degenerate quadratic forms $\mathfrak{J}$ on $U$.

**Proposition 4.1** *If $X_t$ is non-negative strictly $\mathfrak{J}$-separated on $\Lambda_X(U)$ and the associated linear Poincaré flow $P^t$ over any compact invariant subset $\Gamma$ of $\Lambda_X(U)^*$ is strictly $\mathfrak{J}_0$-monotone for some field of quadratic forms $\mathfrak{J}_0$ on $T_\Gamma M$ equivalent to $\mathfrak{J}$, then $\Gamma$ is a hyperbolic set for $P^t$.*

*Proof* The property $\partial_t \mathfrak{J}_0(P^t v) \mid_{t=0} > 0$ for all $v \in N_x$, $x \in \Gamma$ is equivalent to say that the quadratic form $\partial_t \mathfrak{J}_{0,x}|_{N_x}$ is positive definite for all $x \in \Gamma$. This implies the existence of a function $\alpha_1 : U \to (0, +\infty)$ such that

$$\partial_t \mathfrak{J}_0(P^t v) \mid_{t=0} > \alpha_1(x) \cdot \|v\|^2 > 0, \quad x \in \Gamma, v \in N_x, v \neq \vec{0}.$$

Since $\Gamma$ is compact, the smoothness of $\mathfrak{J}_0$ ensures the existence of $\alpha_2 > 0$ such that $|\mathfrak{J}_0(v)| \leq \alpha_2 \|v\|^2$ for all $v \in N_x$, $x \in \Gamma$. Hence we obtain

$$\partial_t \mathfrak{J}_0(P^t v) \mid_{t=0} \geq \alpha_1(x) \cdot \|v\|^2 \geq \frac{\alpha_1(x)}{\alpha_2} |\mathfrak{J}_0(v)|$$

where $\alpha_1(x) > 0$ for all $x \in \Gamma$. Therefore we have

$$\log \frac{\mathfrak{J}_0(P^t v)}{\mathfrak{J}_0(v)} \geq \int_0^t \frac{\alpha_1(X^s(x))}{\alpha_2} \, ds =: H(x, t) \quad \text{for } \mathfrak{J}_0\text{-positive vectors } v; \quad \text{and}$$

$$\log \frac{\mathfrak{J}_0(P^t v)}{\mathfrak{J}_0(v)} \leq -\int_0^t \frac{\alpha_1(X^s(x))}{\alpha_2} \, ds = -H(x, t) \quad \text{for } \mathfrak{J}_0\text{-negative vectors } v.$$

From Lemma 2.18 we can compare $|\mathfrak{J}_0|$ with the square of the Riemannian norm, so all that is left to do is to prove that $H(x, t)$ is unbounded for $t > 0$ and each $x \in \Gamma$. □

**Lemma 4.2** *For every point $x$ in a compact invariant subset $\Gamma \subset \Lambda_X(U)^*$, we have $\lim_{t \to +\infty} H(x, t) = +\infty$.*

*Proof of Lemma 4.2* If there are no singularities in $\Lambda$, then $\Lambda_X(U)^* = \Lambda_X(U)$ is compact and so there exists $\alpha_0 > 0$ such that $\alpha_1(x) \geq \alpha_0$ for all $x \in \Lambda$. This clearly implies the statement of the lemma in this case.

Let $S = \Lambda \cap S(X)$ be the set of finitely many singularities of $X$ in $\Lambda$; we recall that we are assuming that $S$ is formed by hyperbolic fixed points of $X_t$. We fix $\epsilon > 0$ such that $\{B(\sigma, \epsilon)\}_{\sigma \in S}$ is pairwise disjoint sets and $\Lambda \setminus B(S, \epsilon) \neq \emptyset$, where $B(S, \epsilon) = \cup_{\sigma \in S} B(\sigma, \epsilon)$.

We have that $K := \Lambda\backslash B(S,\epsilon)$ is compact and so there exists $\alpha_0 > 0$ such that $\alpha_1(x) \geq \alpha_0$ for all $x \in K$. Moreover, since the norm of the vector field $X$ is bounded from above in $\Lambda$, there exists a minimum time $T > 0$ between consecutive visits of any orbit of $x \in \Lambda\backslash W^s(S)$ to $B(S,\epsilon)$. That is, for $x \in \Lambda\backslash W^s(S)$, if we define sequences $t_1 < s_1 < t_2 < s_2 < \cdots$ such that $X_{(t_i,s_i)}(x) \subset B(S,\epsilon)$ and $X_{[s_i,t_{i+1}]}(x) \subset K$, then $t_{i+1} - s_i > T$, $i \geq 1$.

Since $x \in \Gamma$ and $\Gamma \cap S = \emptyset$, we have $\omega_X(x) \cap B(S,\epsilon) = \emptyset$ for some small $\epsilon > 0$ dependent on $\Gamma$ only, in which case the sequence above terminates at some $s_l$ for $l \geq 1$.

Hence we can write, for all $t \geq s_l$ and $t < t_{l+1}$ (if $t_{l+1}$ does not exist, the last conditions is vacuous)

$$H(x,t) \geq \alpha_0 t_1 + \alpha_0 \sum_{i=1}^{l-1}(t_{i+1} - s_i) + (t - s_l)\alpha_0 \geq \alpha_0[(l-1)T + t_1 + (t - s_l)].$$

Either the sequences are infinite, or $t_{l+1}$ does not exist. Hence $H(x,t)$ grows without bound when $t \to +\infty$. □

Restricting $x$ to a compact invariant subset $\Gamma$ of $\Lambda_X(U)^*$, we obtain

$$\lim_{t\to-\infty} \|P^t|_{N_x^s}\| = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \|P^t|_{N_x^u}\| = +\infty, \quad x \in \Gamma.$$

By well-known results, this ensures that $P^t$ is hyperbolic over $\Gamma$; see e.g. [24]. This concludes the proof. □

### 4.2 Sectional hyperbolicity from the linear Poincaré flow

Here we prove Theorem D through the following results.

Let $\Lambda = \Lambda_X(U)$ be an attracting set contained in the non-wandering set $\Omega(X)$ for a $C^1$ flow given by a vector field $X$. We recall that $\Lambda_X^*(U) = \Lambda_X(U)\backslash \mathrm{Sing}(X)$.

**Theorem 4.3** *The set $\Lambda$ is sectional-hyperbolic for $X$ if, and only if, there is a neighborhood $\mathfrak{U}$ of $X$ in $\mathfrak{X}^1(M)$ such that any compact subset $\Gamma$ of $\Lambda_Y^*(U)$ is hyperbolic of index* $\mathrm{ind}(\Lambda)$ *for the linear Poincaré flow associated to any $Y \in \mathfrak{U}$ and each singularity $\sigma$ of $Y$ in the trapping region $U$ is sectionally hyperbolic with index* $\mathrm{ind}(\sigma) \geq \mathrm{ind}(\Lambda)$.

*Proof* Let us assume that $\Lambda = \Lambda_X(U) \subset \Omega(X)$ is sectionally hyperbolic for $X$. If $E^s \oplus E^c$ is a partially hyperbolic splitting of $TM$ over $\Lambda$, then the projections $N^s := \Pi \cdot E^s$ and $N^u := \Pi \cdot E^c$ are $P^t$-invariant, and $N^s$ is uniformly contracted by $P^t$. Indeed, since the orthogonal projection does not increase norms, for $v \in E_x^s$ we get $\|P^t v\| = \|\Pi_{X_t(x)} DX_t(x) \cdot v\| \leq \|DX_t(x) \cdot v\|$, which is uniformly contracted for $t > 0$, as long as $X(x) \neq \vec{0}$.

Moreover, the above property persists for all vector fields $Y$ in a small enough $C^1$ neighborhood of $X$, by the normal hyperbolic theory; see [15].

Now we assume, additionally, that $E^c$ is sectionally expanding on $\Lambda$ for $X$. This ensures that the continuation $E^{s,Y} \oplus E^{c,Y}$ of the partially hyperbolic splitting for $C^1$ close vector fields is also sectional hyperbolic. For otherwise, according to Remark 1.1, for any given fixed $T > 0$ we would obtain a sequence $Y_n$ of vector fields converging to $X$ in the $C^1$ topology, a sequence $x_n$ of points in $\Lambda_{Y^n}(U)^*$ and a sequence $F_{x_n}$ of 2-subspaces of $E_{x_n}^{c,Y}$ such that $|\det(DY_T^n|_{F_{x_n}})| \leq 2$, $n \geq 1$. Then for a limit point $x$ of $(x_n)_n$ in $\Lambda_X(U)$ and a limit 2-subspace $F_x$ of the sequence $F_{x_n}$ inside $E_x^{c,Y}$ (using the compactness of the Grassmannian over the compact set $\overline{U}$ together with the continuity of the splitting with respect to $Y_n$), we get $|\det(DX_T|_{F_x})| \leq 2$. Since $T > 0$ was arbitrarily chosen, this contradicts the assumption of sectional-expansion on $\Lambda$.

Hence we may argue with any fixed $Y$ close enough to $X$ in the $C^1$ topology. Let us take $\Gamma$ a compact subset of $\Lambda_Y^*$. For $x \in \Gamma$ the uniform expansion along $N^u$ is obtained as follows.

Let $v$ be a unit vector on $N_x^u$ and let $F_x$ be the subspace spanned by $v$ and $X(x)$. For some $K > 0$ we have $K^{-1} \leq \|Y(x)\| \leq K$ for all $x \in \Gamma$ by compactness. Let us fix $t > 0$ and consider the basis $\{\frac{T(x)}{\|T(x)\|},\ v\}$ of $F_x$. We note that $DY_t(F_x)$ is a bidimensional subspace $F_x^t$ of $E^c_{Y_t(x)}$, where we take the basis $\{\frac{Y(Y_t(x))}{\|Y(Y_t(x)x)\|},\ w_t\}$, with

$$w_t := \frac{\Pi_{Y_t(x)} \cdot DY_t(x)(v)}{\|\Pi_{Y_t(x)} \cdot DY^t(x)(v)\|} \quad \text{belonging to} \quad N^u_{Y_t(x)}.$$

With respect to these orthonormal bases we have

$$DY_t|_{F_x} = \begin{bmatrix} \frac{\|Y(Y_t(x))\|}{\|Y(x)\|} & \star \\ 0 & \Delta \end{bmatrix},$$

because the flow direction is invariant. Hence

$$\det\left(DY_t|_{E_x^c}\right) = \frac{\|Y(Y_t(x))\|}{\|Y(x)\|}\Delta \leq K^2\Delta$$

for some $K > 0$ depending only on $\Gamma \subset \Lambda_Y(U)^*$, and

$$\begin{aligned}\|P_x^{Y,t} \cdot v\| &= \|\Pi_{X_t(x)} \cdot DY_t(x)(v)\| = \|\Delta \cdot w\| = |\Delta| \\ &\geq K^{-2}|\det(DY_t|_{F_x})| \geq K^{-2}Ce^{\lambda t}.\end{aligned}$$

This proves that $N^u$ is uniformly expanded by the linear Poincaré flow $P^{Y,t}$ over $\Gamma$. Moreover, for every singularity $\sigma \in \Lambda$ we have $T_\sigma M = E^s_\sigma \oplus E^u_\sigma$ a sectional hyperbolic splitting, thus $\mathrm{ind}(\sigma) \geq \mathrm{ind}(\Lambda)$; in fact, sectional expansion on $E^c_\sigma$ ensures that either $\mathrm{ind}(\sigma) = \mathrm{ind}(\Lambda)$ or $\mathrm{ind}(\sigma) = \mathrm{ind}(\Lambda) + 1$.

Reciprocally, let us assume that $\Lambda$ is a compact attracting set with isolating neighborhood $U$ such that: the linear Poincaré flow over any compact subset $\Gamma \subset \Lambda_Y^*(U)$ is hyperbolic with constant index $\mathrm{ind}(\Lambda)$, for all $Y$ in a $C^1$ neighborhood $\mathfrak{U}$ of $X$; and that the singularities $\sigma$ in $U$ for each $Y \in \mathfrak{U}$ are sectionally hyperbolic with index $\mathrm{ind}(\sigma) \geq \mathrm{ind}(\Lambda)$. In particular, the index of all periodic orbits of $U$ for $Y \in \mathfrak{U}$ is constant $\mathrm{ind}(\Lambda)$, and the flows in $\mathfrak{U}$ are homogeneous. Hence, every periodic point $p$ in $U$ for $Y$ is hyperbolic with uniform bounds on the expansion and contraction on the period and, moreover, admits a sectional-hyperbolic splitting $T_pM = E_p^{Y,s} \oplus E_p^{Y,c}$ of the tangent bundle with constant index $\mathrm{ind}(\Lambda)$ and with angle between the stable and central directions uniformly bounded away from zero; see [5, Section 5.4.1].

This is enough to deduce that the tangent bundle on $\Lambda_Y(U) \cap \Omega(X)$ admits a partially hyperbolic splitting $E^s \oplus E^c$ with index $\dim E^s = \mathrm{ind}(\Lambda)$. Indeed, for a non-singular $x \in \Lambda_Y(U)$ we can use Shub's Closing Lemma to obtain sequences $Y^k \in \mathfrak{U}$ and $p_k \in U$ a periodic point for $Y^k$ such that $Y^k \xrightarrow{C^1} Y$ and $p_k \to x$. We then define $E_x^{s,Y} = \lim_{k\to\infty} E_{p_k}^{s,Y^k}$ and $E_x^{c,Y} = \lim_{k\to\infty} E_{p_k}^{c,Y^k}$. This decomposition will be $DY_t$-invariant and partially hyperbolic by construction. Moreover the assumption on the index of the singularities ensures that the partial hyperbolic splitting on every periodic orbit for each flow $Y \in \mathfrak{U}$ can be extended to a partially hyperbolic splitting on the entire $\Lambda$, including the singularities; see [5, Section 5.4.2]. Having these properties robustly on $\mathfrak{U}$ with sectional hyperbolicity on periodic orbits implies that the subbundle $E^c$ is sectionally expanding, for $Y \in \mathfrak{U}$; see [5, Section 5.4.3]. This completes the proof. $\square$

Now Proposition 4.1 shows that strict $\mathcal{J}$-monotonicity for the linear Poincaré flow over a compact invariant subset implies hyperbolicity. Together with Theorem 4.3 we conclude the proof of sufficiency in Theorem D.

This completes the proof that strict $J$-monotonicity of the linear Poincaré flow implies sectional hyperbolicity, which is half of the statement of Theorem D.

### 4.3 Strict $\mathcal{J}$-monotonicity for the linear Poincaré flow

Now we prove that having a sectional hyperbolic splitting implies that there exists a field of non-degenerate and indefinite quadratic forms $\mathcal{J}$ with constant index equal to the dimension of the contracting direction, such that the linear Poincaré flow is strictly $\mathcal{J}_0$-monotonous on every compact invariant set without singularities, for some compatible field of forms $\mathcal{J}_0$, completing the proof of Theorem D.

Let $\Lambda = \Lambda_X(U)$ be a compact maximal invariant set admitting a sectional hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$. As noted in the first part of the proof of Theorem 4.3, the existence of sectional hyperbolic splitting is a robust property: there exists a neighborhood $\mathcal{U}$ of $X$ in the $C^1$ topology in $\mathfrak{X}^1(M)$ such that all $Y \in \mathcal{U}$ have a maximal invariant subset $\Lambda_Y(U)$ which is also sectional hyperbolic. Hence the results we obtain below hold robustly in a neighborhood of $X$.

We have already shown, in Sect. 2.5, how to construct a field of quadratic forms $\mathcal{J}$ such that $X$ is strictly $\mathcal{J}$-separated on a neighborhood $V \subset U$ of $\Lambda$ satisfying for some $\lambda > 0$ and all $x \in \Lambda$ and $t > 0$

$$|DX_t v^+| = \sqrt{\mathcal{J}(DX_t v^+)} \ge e^{\lambda t}\sqrt{\mathcal{J}(DX_t v^-)} = e^{\lambda t}|DX_t v^-|, \quad v^- \in E^s_x, v^+ \in E^c_x, |v^\pm| = 1.$$

The results in [14] extend the properties of adapted metrics to partial hyperbolic splittings, in such a way that we can also obtain for all $t > 0$

$$|DX_t v^-| \le e^{-\lambda t}, \quad v^- \in E^s_x, |v^-| = 1.$$

On $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$ we define $N^s = \prod \cdot E^s$ and $N^u = \prod \cdot E^c$, where $\prod$ is the projection of the tangent bundle onto the pseudo-orthogonal complement $N$ of $X$ with respect to $\mathcal{J}$. We note that since $P^t = \prod \cdot DX_t$

$$|P^t \mid_{N^s_x}| \cdot |P^{-t} \mid_{N^u_{X_t(x)}}| \le |DX_t \mid_{E^s_x}| \cdot |DX_{-t} \mid_{E^c_{X_t(x)}}| \le e^{-\lambda t}, \quad \text{and} \tag{4.1}$$

$$|P^t \mid_{N^s_x}| \le |DX_t \mid_{E^s_x}| \le e^{-\lambda t}, \quad x \in \Lambda^*, t > 0 \tag{4.2}$$

so that the linear Poincaré flow has a partially hyperbolic splitting over $\Lambda^*$.

The assumption of sectional expansion ensures that, if we fix any unit vector $v \in N^u_x$ for $x \in \Lambda^*$, then for some $C, \lambda > 0$ and every $t > 0$

$$\begin{aligned} Ce^{\lambda t} &\le |\det DX_t \mid_{\mathrm{span}\{X(x), v\}}| = \frac{\mathrm{vol}(DX_t v, X(X_t(x)))}{\mathrm{vol}(X(x), v)} \\ &= \frac{|X(X_t(x))|}{|X(x)|}|DX_t v| \sin \angle(DX_t v, X(X_t(x))) = \frac{|X(X_t(x))|}{|X(x)|}|P^t v|. \end{aligned}$$

Since we are in a compact set we have $0 < c_0 = \sup_{z\in\Lambda} |X(z)| < \infty$ and so

$$|P^t v| \ge \frac{C|X(x)|}{c_0} e^{\lambda t}, \quad x \in \Lambda^*, v \in N^u_x, |v| = 1, t > 0. \tag{4.3}$$

We write $c(x) := C|X(x)|/c_0$ and note that $0 < c(x) \le 1$ by letting $t \to 0$ in the above inequality.

We restrict now to the case of a compact invariant subset $\Gamma$ of $\Lambda^*$. In this case $c(x) \geq c_1 > 0$ for all $x \in \Gamma$ and $N^s_\Gamma \oplus N^u_\Gamma$ is a uniformly hyperbolic splitting for $P^t$. We can then obtain an adapted Riemannian metric for this splitting following [14] and define a field $\mathcal{J}_0$ of quadratic forms using this adapted metric as in Sect. 2.5.1. With respect to the adapted metric we obtain both (4.1) and (4.2), and also (4.3) but with unit constants multiplying the exponential. From Remark 2.22, since $\mathcal{J}$ and $\mathcal{J}_0$ have the same signs on the $E^s_\Gamma$ and $E^c_\Gamma$, then $\mathcal{J}_0 \sim \mathcal{J}$.

We show that $P^t$ is strictly $\mathcal{J}_0$-monotonous. We consider a vector $v \in N_x$ with $v = v^- + v^+$, $v^- \in N^s_x$, $v^+ \in N^u_x$ and $\mathcal{J}_0(v^+) - \mathcal{J}_0(v^-) = |v^+|^2 + |v^-|^2 = 1$ for $x \in \Gamma$, and the norm of its image under the linear Poincaré flow. Since $P^t v = P^t v^+ + P^t v^-$ and $N^s$ and $N^u$ are $P^t$-invariant, if $v^+ \neq \vec{0}$ we get for $t > 0$

$$\begin{aligned}\mathcal{J}_0(P^t v) = \mathcal{J}_0(P^t v^+) + \mathcal{J}_0(P^t v^-) &= \mathcal{J}_0(P^t v^+) \cdot \left(1 + \frac{\mathcal{J}_0(P^t v^-)}{\mathcal{J}_0(P^t v^+)}\right)\\ &\geq e^{2\lambda t}\mathcal{J}_0(v^+)\left(1 + e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right),\end{aligned}$$

since $\mathcal{J}(P^t v^-) < 0$. We note that the value of the left hand side and the right hand side above are the same at $t = 0$. Moreover the derivative of the right hand side with respect to $t$ at $t = 0$ equals

$$\left[2\lambda e^{2\lambda t}\mathcal{J}_0(v^+)\left(1 + e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right) - e^{2\lambda t}\mathcal{J}_0(v^+) \cdot 2\lambda e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right]\Bigg|_{t=0} = 2\lambda\mathcal{J}_0(v^+) > 0.$$

Hence we conclude that $\partial_t \mathcal{J}_0(P^t v) \mid_{t=0} \geq 2\lambda\mathcal{J}_0(v^+) > 0$ when $v$ has a non-zero positive component. In the remaining case, $v = v^-$ we obtain (again because $\mathcal{J}_0(P^t v^-) < 0$)

$$\mathcal{J}_0(P^t v^-) \geq e^{-2\lambda t}\mathcal{J}_0(v^-)$$

with the same value at $t = 0$ on both sides, so that $\partial_t \mathcal{J}_0(P^t v^-) \mid_{t=0} \geq -2\lambda\mathcal{J}_0(v^-) > 0$ also in this case.

We have proved that for all vector fields $Y$ sufficiently $C^1$ close to $X$ and for every compact invariant subset $\Gamma$ of $\Lambda_Y(U)^*$, we can find a field $\mathcal{J}_0$ of quadratic forms compatible to $\mathcal{J}$ over $\Gamma$ such that $P^t_Y$ is strictly $\mathcal{J}_0$-monotonous, as claimed in Theorem D.

This together with Theorem 4.3 completes the proof of Theorem D.

### 4.4 Criteria for $\mathcal{J}$-monotonicity of the linear Poincaré flow

Here we prove Proposition 1.4. We have already characterized partial hyperbolicity using the notion of $\mathcal{J}$-separation, or the existence of infinitesimal Lyapunov functions, which depend only on the vector field $X$ and its derivative $DX$. To present a characterization of sectional hyperbolicity along the same lines, we must use the conclusion of Theorem D, and obtain a criterion for the linear Poincaré flow to be $\mathcal{J}$-monotonous.

The condition of $\mathcal{J}$-monotonicity for the linear Poincaré flow can be expressed using only the vector field $X$ and its space derivative $DX$.

Recall that a self-adjoint operator is said to be (positive) non-negative if all eigenvalues are (positive) non-negative.

**Lemma 4.4** *Let $X$ be a $\mathcal{J}$-non-negative (positive) vector field on $U$. Then, the linear Poincaré flow is (strictly) $\mathcal{J}$-monotone if, and only if, the operator*

$$\hat{J}_x := DX(x)^* \cdot \Pi^*_x J \Pi_x + \Pi^*_x J \Pi_x \cdot DX(x)$$

*is a non-negative (positive) self-adjoint operator.*

Here, we consider $\Pi^*$ as the adjoint operator of the orthogonal projection $\Pi$ in the definition of the linear Poincaré flow.

The conditions above are again consequence of the corresponding results for linear multiplicative cocycles over flows, as explained in Sect. 4.

*Proof* We shall prove only the positive case, once the non-negative is similar.

We denote by $|v| := \langle Jv, v\rangle^{1/2}$ the $\mathcal{J}$-norm of a vector $v$ and observe that we can write

$$\Pi_{X_t(x)} v := v - \langle Jv, \frac{X(X_t(x))}{|X(X_t(x))|}\rangle \frac{X(X_t(x))}{|X(X_t(x))|},$$

for all $v \in T_x M$ with $X(X_t(x)) \neq \vec{0}$ and $t \geq 0$. Then, to conclude that $\mathcal{J}(P^t v) > \mathcal{J}(v)$ for every $v \in N_x$, $x \in U$, $X(x) \neq \vec{0}$ it is enough to prove

$$\partial_t \mathcal{J}(P^t v) > 0 \quad \text{for every} \quad v \in T_x M,\ X(X_t(x)) \neq \vec{0} \quad \text{and} \quad t \geq 0. \tag{4.4}$$

Reciprocally, if we have that $\mathcal{J}(P^t v) > \mathcal{J}(v)$ for every $v \in N_x$, $x \in U$, $X(x) \neq \vec{0}$, then we also must have (4.4).

Now the above derivative can be written, just like in the previous sections

$$\langle J \cdot \Pi_{X_t(x)} DX_t v, \partial_t(\Pi_{X_t(x)} DX_t v)\rangle + \langle J \cdot \partial_t(\Pi_{X_t(x)} DX_t v), \Pi_{X_t(x)} DX_t v\rangle. \tag{4.5}$$

To expand the above expression, we note that

$$\partial_t(\Pi_{X_t(x)} DX_t v) = \partial_t \left( DX_t v - \left\langle J \cdot DX_t v, \frac{X(X_t(x))}{|X(X_t(x))|} \right\rangle \frac{X(X_t(x))}{|X(X_t(x))|} \right)$$

can be written in the following way,

$$\begin{aligned} &DX(X_t(x))DX_t v + \langle J \cdot DX_t v, \frac{X(X_t(x))}{|X(X_t(x))|}\rangle \cdot \partial_t \frac{X(X_t(x))}{|X(X_t(x))|} \\ &- \left( \left\langle J \cdot DX(X_t(x))DX_t v, \frac{X(X_t(x))}{|X(X_t(x))|} \right\rangle + \left\langle J \cdot DX_t v, \partial_t \frac{X(X_t(x))}{|X(X_t(x))|} \right\rangle \right) \frac{X(X_t(x))}{|X(X_t(x))|}. \end{aligned}$$

Since $\partial_t \frac{X(X_t(x))}{|X(X_t(x))|}$ equals

$$-\left\langle \frac{X(X_t(x))}{|X(X_t(x))|}, DX(X_t(x)) \frac{X(X_t(x))}{|X(X_t(x))|} \right\rangle \cdot \frac{X(X_t(x))}{|X(X_t(x))|} + DX(X_t(x)) \frac{X(X_t(x))}{|X(X_t(x))|}$$

and must be $\mathcal{J}$-orthogonal to the flow direction at $X_t(x)$, then this last expression is the projection on $N_{X_t(x)}$ as follows

$$\partial_t \frac{X(X_t(x))}{|X(X_t(x))|} = \left(\Pi_{X_t(x)} DX(X_t(x))\right) \cdot \frac{X(X_t(x))}{|X(X_t(x))|}$$

Now replacing $X_t(x)$ by $z$ throughout and the vector $X(z)$ $\mathcal{J}$-normalized by $\hat{X}(z)$ we obtain the following expression for the derivative of $P^t v$

$$\begin{aligned} &DX(z)DX_t v - \langle J \cdot DX_t v, \hat{X}(z)\rangle \cdot \Pi_z DX(z)\hat{X}(z) \\ &-(\langle J \cdot DX(z)DX_t v, \hat{X}(z)\rangle + \langle J \cdot DX_t v, \Pi_z DX(z)\hat{X}(z)\rangle)\hat{X}(z) \end{aligned}$$

or, easier for a geometrical interpretation

$$\begin{aligned} &DX(z)DX_t v - \langle J \cdot DX(z)DX_t v, \hat{X}(z)\rangle \hat{X}(z) \\ &-\langle J \cdot DX_t v, \hat{X}(z)\rangle \cdot \Pi_z DX(z)\hat{X}(z) - \langle J \cdot DX_t v, \Pi_z DX(z)\hat{X}(z)\rangle \hat{X}(z). \end{aligned}$$

We observe that the first line above is the projection on $N_z$ of $DX(z)DX_t v$ so we have that $\partial_t P^t v$ equals

$$\Pi_z DX(z)DX_t v - \langle J \cdot DX_t v, \hat{X}(z)\rangle \Pi_z DX(z)\hat{X}(z) - \langle J \cdot DX_t v, \Pi_z DX(z)\hat{X}(z)\rangle \hat{X}(z).$$

In the expression (4.5), we take the $\mathcal{J}$-(inner)-product with a vector on $N_z$, so the $\hat{X}$ component above contributes nothing to the final result. Therefore (4.5) becomes

$$\begin{aligned}&\langle J \cdot \Pi_z DX_t v, \Pi_z DX(z)DX_t v\rangle + \langle J \cdot \Pi_z DX(z)DX_t v, \Pi_z DX_t v\rangle \\ &\quad -\langle J \cdot DX_t v, \hat{X}(z)\rangle(\langle J \cdot \Pi_z DX_t v, \Pi_z DX(z)\hat{X}(z)\rangle + \langle J \cdot \Pi_z DX(z)\hat{X}(z), \Pi_z DX_t v\rangle)\end{aligned}$$

and using the adjoint of $DX(z) = DX(X_t(x))$ we define $\hat{J} = \hat{J}_x := \left(\Pi_x DX(x)\right)^* J \Pi_x + \Pi_x^* J \Pi_x DX(x)$ and obtain

$$\partial_t P^t v = \langle \hat{J}_{X_t(x)} DX_t v, DX_t v\rangle - \langle J \cdot DX_t v, \hat{X}(X_t(x))\rangle \cdot \langle \hat{J} \cdot DX_t v, \hat{X}(X_t(x))\rangle.$$

Letting $t = 0$, since $\langle J \cdot v, \hat{X}(x)\rangle = 0$ for $v \in N_x$, we arrive at

$$\partial_t \left(P^t v\right) |_{t=0} = \langle \hat{J}_x v, v\rangle - \langle J \cdot v, \hat{X}(x)\rangle \cdot \langle \hat{J} \cdot v, \hat{X}(x)\rangle = \langle \hat{J}_x v, v\rangle. \tag{4.6}$$

We conclude that condition (4.4) is equivalent to

$$\langle \hat{J}_{X_t(x)} v, v\rangle > 0, \quad v \in N_{X_t(x)}, \tag{4.7}$$

that is, $\hat{J}_x$ *is a positive definite self-adjoint operator on* $N_x$ *for each* $x \in U$ *with* $X(x) \neq \vec{0}$. Indeed, by the flow property of $P^t$ we have, for all $s > 0$

$$\partial_t \mathcal{J}(P^{t+s} v) |_{t=0} = \partial_t \mathcal{J}\left(P^t (P^s v)\right) |_{t=0} > 0 \quad \text{because} \quad P^s v \in N_{X^s(x)}$$

and $P^s \colon N_x \to N_{X^s(x)}$ is an isomorphism. □

So (4.7) is the condition that the vector field and its derivative must satisfy in $U$, except at singularities, so that the linear Poincaré flow admits a hyperbolic splitting.

This concludes the proof of Proposition 1.4.

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