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J. Differential Equations 257 (2014) 3165–3199

Journal of Differential Equations

www.elsevier.com/locate/jde

Fourth order evolution equations which describe pseudospherical surfaces

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Received 10 December 2013; revised 17 April 2014

Available online 4 July 2014

Abstract

Differential equations that describe pseudospherical surfaces are considered. These equations are equivalent to the structure equations of a metric with Gaussian curvature $K = -1$. They can also be described as the compatibility condition of an associated linear problem also referred to as a zero curvature representation. A complete and explicit classification of a class of fourth order evolution equations is given. The classification provides four huge classes (referred to as Types I–IV) of fourth order evolution equations that describe pseudospherical surfaces, together with the associated one (or more) parameter linear problems. The differential equations of each type are determined by choosing certain arbitrary differentiable functions. Fourth-order member of the Burgers hierarchy and a modified Kuramoto–Sivashinsky equation are examples of equations described by Types I and IV, respectively. Many other explicit examples are presented.

MSC: 35G20; 47J35; 53C21; 53B20

Keywords: Fourth order evolution equations; Pseudospherical surfaces; Nonlinear partial differential equations

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[1] Partially supported by Ministry of Science and Technology, Brazil (CNPq grant 303774/2009-6).

http://dx.doi.org/10.1016/j.jde.2014.06.010

1. Introduction

Differential equations which describe pseudospherical surfaces appear in a wide range of contexts, from physics to applied and pure mathematics, as suitable models in the description of nonlinear phenomena. Geometrically these equations are characterized by the fact that their solutions provide metrics on open subsets of $\mathbb{R}^2$, with Gaussian curvature $K = -1$.

The first well known example of such an equation is the sine-Gordon equation $z_{xt} = \sin(z)$. This example was discovered by Edmond Bour [7], who realized that the Gauss–Mainardi–Codazzi equations for pseudospherical surfaces contained in $\mathbb{R}^3$, in terms of Darboux asymptotic coordinates, reduce to the sine-Gordon equation. The discovery of Bäcklund transformations [2] first, and later the construction by Bianchi of the superposition formula for solutions of this equation [8], focused even more attention on the sine-Gordon equation, that turned out to be an important model in the description of several nonlinear phenomena (see for example [19,21,33]).

The interest on the class of differential equations that describe pseudospherical surfaces began with the early observation [32] that equations such as KdV, MKdV and the sine-Gordon equation belong to this class. The definition and a theoretical approach in studying such equations were introduced in the fundamental paper by Chern and Tenenblat [12]. The importance of the differential equations that describe pseudospherical surfaces is due to the fact that, as a consequence of its definition, such a differential equation is the integrability condition of a linear system of partial differential equations. This linear system may be used in the inverse scattering method to provide solutions to the differential equation. Moreover, more recent studies on nonlinear phenomena described by such equations (see for example [9,15,26,29]) prove the relevance of these equations and justify our general interest in their study and their classification.

A differential equation for a real function $z(x,t)$ is said to describe pseudospherical surfaces if is equivalent to the structure equations, $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$, of a 2-dimensional Riemannian manifold whose Gaussian curvature $K = -1$, with 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ where f_{ij} are smooth functions of z and its derivatives.

In [12], the authors obtained classification results for evolution equations of the form $z_t = F(z, z_1, ..., z_k)$ (from now on we denote $z_i = \partial^i z/\partial x^i$), with the assumption that $f_{21} = \eta$, where η is a parameter. A similar problem for equations of the form $z_{xt} = F(z, z_1, ..., z_k)$ was also considered. A noteworthy result of this study was an effective method for the explicit determination of entire new classes of differential equations that describe pseudospherical surfaces. Motivated by the results in [12], in a series of subsequent papers [18,22–24], this method was sistematically implemented and new classes of pseudospherical equations were identified still with the assumption that $f_{21} = \eta$. In [10], the authors showed how the geometric properties of pseudospherical surfaces may provide infinite number of conservation laws when the functions f_{ij} are analytic functions of the parameter η. The parameter appearing in the linear problem is important not only in the existence of infinite number of conservation laws, but is also related to Bäcklund transformations and to the inverse scattering method [5].

In 1995, Kamran and Tenenblat [20] gave a complete characterization of evolution equations of type $z_t = F(z, z_1, ..., z_k)$ which describe pseudospherical surfaces, in terms of necessary and sufficient conditions that have to be satisfied by F and the functions f_{ij}, with no restriction on the functions f_{ij}.

Reyes [25] considered evolution equations of the more general form $z_t = F(x, t, z, z_1, ..., z_k)$, allowing x, t to appear explicitly in the equation, with the assumption that $f_{21} = \eta$. Then, in a subsequent series of papers [26–28] Reyes also studied other aspects of such equations.

Differential systems describing pseudospherical surfaces or spherical surfaces (metrics with constant positive curvature) were studied by Ding and Tenenblat in 2002 [14]. Such systems include equations such as the nonlinear Schrödinger equation and the Heisenberg Ferromagnet model. Large new families of differential systems describing pseudospherical surfaces were obtained. These families intersect those obtained by Fokas in [15].

The complete characterization results obtained in [20] are extremely useful, either in checking if a given differential equation describes pseudospherical surfaces or in generating large families of such equations. As an application of [20], more recently in [17], Gomes Neto gave a classification of fifth order evolution equations of the form $z_t = z_5 + G(z, z_1, z_2, z_3, z_4)$, under the assumption that f_{21} and f_{31} are linear combination of f_{11}. This assumption introduces four parameters and it extends the case previously studied $f_{21} = \eta$. The results in [17] permitted the explicit description of huge classes of such equations.

All the above mentioned results produced, apart from the already well known examples of differential equations that describe pseudospherical surfaces, a great amount of new equations whose physical relevance is highly expected. For example, some applications of equations classified by Rabelo and Tenenblat [18,22–24] have been recently suggested (see for example [29–31]). However, the same should occur, for instance, in the case of results presented in [17] and in this paper.

We should mention that a higher dimensional geometric generalization of the sine-Gordon equation, characterizing n-dimensional submanifolds of the Euclidean $\mathbb{R}^{2n-1}$ with constant sectional curvature $K = -1$, was considered in [35] and its intrinsic version as a metric on open subsets of $\mathbb{R}^n$, with $K = -1$, was studied in [6]. Other differential n-dimensional systems that are the integrability condition of linear systems of PDEs can be found in the so called *generating system* (see [34] and its references).

In this paper, we classify differential equations that describe pseudospherical surfaces of the form

$$z_t = z_4 + G(z, z_1, z_2, z_3), \tag{1.1}$$

with associated 1-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \qquad \omega_2 = f_{21}dx + f_{22}dt, \qquad \omega_3 = f_{31}dx + f_{32}dt, \tag{1.2}$$

with the assumption, analogue to that in [17], that

$$f_{s1} = \mu_s f_{11} + \eta_s, \quad \mu_s, \eta_s \in \mathbb{R}, \ 2 \le s \le 3. \tag{1.3}$$

The reason for such an assumption, besides extending the case $f_{21} = \eta$, is that it has the advantage of simplifying noteworthily the classification of differential equations that describe pseudospherical surfaces of the form (1.1). Our interest in fourth order equations is due to the fact that very little is known for such equations in the existing literature.

Our main result shows that the evolution equations that we are considering are classified into four large types of equations. In each type, a differential equation is obtained by choosing arbitrarily certain differentiable functions. Fourth-order member of the Burgers hierarchy and a modified Kuramoto–Sivashinsky equation are examples of such equations. Many other examples are presented as well. Each type is presented in a form which permits to identify the one (or more)-parameter linear problem, whose integrability condition is the evolution equation.

The paper is organized as follows. In Section 2, we collect some preliminaries on differential equations that describe pseudospherical surfaces. Moreover, given such an equation with associated 1-forms ω_i, we provide the linear system of PDEs whose integrability condition is the differential equation.

In Section 3, we state our main result (Theorem 3.1), which classifies the differential equations into Types I–IV and we summarize the classification scheme followed in the subsequent Sections 5 and 6. Moreover, we give some simple examples from the huge classes of equations described in the main theorem.

Section 4 is devoted to a preliminary analysis which will be used in the rest of the paper. In particular, it is observed that the classification is based on the separate analysis of the two main cases $\rho \neq 0$ and $\rho = 0$ (where ρ is a constant determined by the parameters μ_s, η_s). These two cases will be separately analyzed in Section 5 and Section 6, respectively.

Section 5 gives a complete description of the differential equations that describe pseudospherical surfaces corresponding to $\rho \neq 0$. This case is completely studied in Theorems 5.3, 5.5 and 5.6, which provide the differential equations of Types I, II and III respectively. In each theorem, the associated 1-forms corresponding to each type of equation are given explicitly.

Section 6 gives a complete description of differential equations that describe pseudospherical surfaces corresponding to $\rho = 0$. As in the previous section, this case is completely studied and a complete description of the 1-forms ω_i, together with the corresponding pseudospherical equation, is provided in Theorems 6.2, 6.4 and 6.5. The differential equations of Type IV are given in Theorem 6.4. Those given in Theorems 6.2 and 6.5 are of Types I and III respectively, however the associated 1-forms may be different from the ones given in Theorems 5.3 and 5.6 in Section 5.

In Section 7 we prove the main theorem, Theorem 3.1, as an immediate consequence of the results of the previous Sections 5 and 6. Moreover, we provide other examples of equations obtained from our classification. In particular, it is shown that the fourth-order member of the Burgers hierarchy and a modified Kuramoto–Sivashinsky equation are differential equations of Types I and IV, respectively.

2. Preliminaries

If (M, g) is a 2-dimensional Riemannian manifold and $\{\omega_1, \omega_2\}$ is a coframe, dual to an orthonormal frame $\{e_1, e_2\}$, then $g = \omega_1^2 + \omega_2^2$ and ω_i satisfy the structure equations: $d\omega_1 = \omega_3 \wedge \omega_2$ and $d\omega_2 = \omega_1 \wedge \omega_3$, where ω_3 denotes the connection form defined as $\omega_3(e_i) = d\omega_i(e_1, e_2)$. The Gaussian curvature of M is the function K such that $d\omega_3 = -K\omega_1 \wedge \omega_2$.

Now, a differential equation $\mathcal{E}$, for a real-valued function $z(x,t)$, *describes pseudospherical surfaces* if it is equivalent to the structure equations of a surface with Gaussian curvature $K = -1$, i.e.,

$$d\omega_1 = \omega_3 \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_3, \qquad d\omega_3 = \omega_1 \wedge \omega_2, \tag{2.1}$$

where $\{\omega_1, \omega_2, \omega_3\}$ are 1-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \qquad \omega_2 = f_{21}dx + f_{22}dt, \qquad \omega_3 = f_{31}dx + f_{32}dt, \tag{2.2}$$

with $\omega_1 \wedge \omega_2 \neq 0$ and f_{ij} are functions of $z(x,t)$ and derivatives of $z(x,t)$.

The classical celebrated example of the sine-Gordon equation $z_{xt} = \sin(z)$ corresponds to

$$\omega_1 = \frac{1}{\eta}\sin(z)\,dt, \qquad \omega_2 = \eta dx + \frac{1}{\eta}\cos(z)\,dt, \qquad \omega_3 = z_x\,dx.$$

Equations that describe pseudospherical surfaces can also be characterized in a different way. For example, the system of equations (2.1) is equivalent to the integrability condition of the linear system

$$\begin{pmatrix} dv^1 \\ dv^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{2.3}$$

where $v^i = v^i(x,t)$. Moreover, by using the notation $V = (v^1, v^2)^T$ and considering the $sl(2,\mathbb{R})$-valued smooth functions (also known as Lax pair in matrix form)

$$A = \frac{1}{2}\begin{pmatrix} f_{21} & f_{11} - f_{31} \\ f_{11} + f_{31} & -f_{21} \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} f_{22} & f_{12} - f_{32} \\ f_{12} + f_{32} & -f_{22} \end{pmatrix}, \tag{2.4}$$

(2.3) can be written as the linear problem

$$\frac{\partial V}{\partial x} = AV, \qquad \frac{\partial V}{\partial t} = BV. \tag{2.5}$$

The linear system (2.5) or (2.3) is usually referred to as *the linear problem associated to* $\mathcal{E}$. An important contribution to solving the differential equation $\mathcal{E}$, is the application of the inverse scattering method to the linear problem associated to $\mathcal{E}$. This method was introduced in [16] and it can be reformulated in terms of a Riemann–Hilbert factorization problem [3,4]. In particular, when $f_{21} = \eta$, where η is a parameter and f_{11}, f_{31} are independent of η, the linear problem (2.5) is the so called AKNS system [1].

It is easy to show that (2.1) is equivalent to the integrability condition of (2.5), namely

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A, B] = 0. \tag{2.6}$$

This condition is usually referred to as an $sl(2,\mathbb{R})$-valued *zero-curvature representation* [13] for the differential equation $\mathcal{E}$. Notice that, (2.6) is invariant under the gauge transformations: $A \to A^S = SAS^{-1} + S_{,x}S^{-1}$, $B \to B^S = SBS^{-1} + S_{,t}S^{-1}$, where S is an $SL(2,R)$-valued smooth function.

Throughout this paper, partial derivatives of z of order i with respect to x will be denoted by z_i, i.e.,

$$z_i = \frac{\partial^i z}{\partial x^i}.$$

Hence, an evolution equation of order k, not depending on x and t explicitly, will be written in the form

$$z_t = F(z, z_1, ..., z_k). \tag{2.7}$$

Necessary and sufficient conditions for equation (2.7) to describe pseudospherical surfaces are given by the following theorem

Theorem 2.1. *(See [20].) An equation*

$$z_t = F(z, z_1, \dots, z_k)$$

describes pseudospherical surfaces if, and only if, there exist functions f_{ij} *such that*

$$f_{11}f_{22} - f_{12}f_{21} \neq 0, \tag{2.8}$$

$$\Delta = (f_{11,z})^2 + (f_{21,z})^2 + (f_{31,z})^2 \neq 0, \tag{2.9}$$

$$f_{i1,z_j} = 0, \qquad f_{i2,z_k} = 0, \quad 1 \le i \le 3,\ 1 \le j \le k, \tag{2.10}$$

satisfying differential equations

$$\begin{aligned} &f_{21}f_{32} - f_{22}f_{31} - f_{11,z}F + B_{12} = 0,\\ &f_{12}f_{31} - f_{11}f_{32} - f_{21,z}F + B_{22} = 0,\\ &f_{12}f_{21} - f_{11}f_{22} - f_{31,z}F + B_{32} = 0, \end{aligned} \tag{2.11}$$

where

$$B_{j2} = \sum_{i=0}^{k-1} f_{j2,z_i} z_{i+1}, \qquad 1 \le j \le 3. \tag{2.12}$$

In order to obtain classification results of differential equations that describe pseudospherical surfaces of the form (2.7) one has to obtain F and the functions f_{ij} satisfying (2.8)–(2.11).

3. Main result and simple examples

In this section, we state our main theorem that provides a complete classification of the differential equations that describe pseudospherical surfaces of the form (1.1), under the assumption that the associated 1-forms, $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \le i \le 3$, satisfy (1.3), i.e., $f_{s1} = \mu_s f_{11} + \eta_s$, $\mu_s, \eta_s \in \mathbb{R}$, $s = 2, 3$. Moreover, we provide some examples of such evolution equations. The proof of the main theorem will be given in the following sections.

In order to describe the classification scheme, we introduce the following notation for some constants defined in terms of μ_s and η_s, that will be used throughout this paper:

$$\begin{aligned} a &= \mu_2\eta_2 - \mu_3\eta_3,\\ \alpha &= 1 + \mu_2^2 - \mu_3^2,\\ \rho &= -\eta_3\alpha + 2\mu_3 a,\\ \delta_1 &= -\alpha\big(\eta_2^2 - \eta_3^2\big) + a^2,\\ \delta_2 &= -\mu_2^2\big(1 - \mu_3^2\big) - \big(1 + \mu_3^2\big)^2,\\ \gamma &= \mu_2(\mu_3\eta_2 - \eta_3\mu_2) - \eta_3. \end{aligned} \tag{3.1}$$

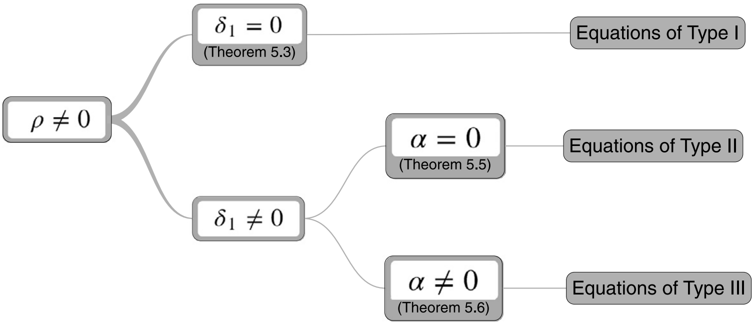

Fig. 3.1. Classification scheme when $\rho \neq 0$.

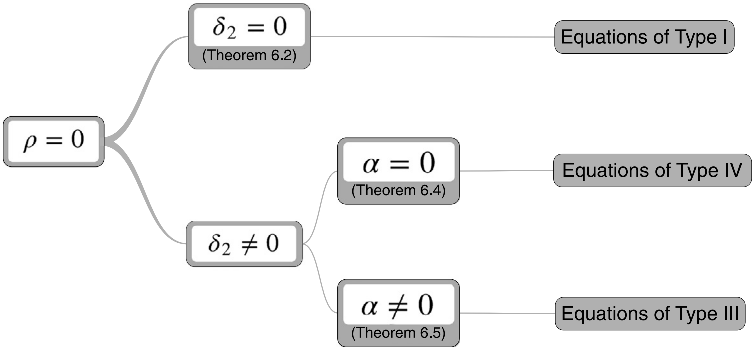

Fig. 3.2. Classification scheme when $\rho = 0$.

In Section 4, we will show that the classification naturally splits into two main cases: $\rho \neq 0$ and $\rho = 0$, where ρ is the constant defined by (3.1). These two cases will be analyzed separately in Section 5 and Section 6, respectively. In Section 5, we will show that equations corresponding to $\rho \neq 0$ can be subdivided into three types as shown in Fig. 3.1, where α and δ_1 are given in (3.1). Analogously, in Section 6, we will show that equations corresponding to $\rho = 0$ can be subdivided into 3 cases as shown in Fig. 3.2, where δ_2 and α are given in (3.1), providing only one new type of equation. Each one of the four types of equations is given explicitly in terms of certain arbitrary differentiable functions. Figs. 3.1 and 3.2 summarize the classification scheme of the paper and they can be used as a reference to identify the type of a given equation.

The results of Sections 5 and 6, will prove the following theorem which is the main result of the paper

Theorem 3.1. *An evolution equation which describes pseudospherical surfaces of the form (1.1), with associated* 1*-forms given by (1.2) satisfying (1.3), belongs to one of the following four types:*

Type I:

$$z_t = z_4 + \left[\frac{2h''}{h'} z_1 + h + \frac{\psi_{,z_1}}{h'}\right] z_3 + \left(\frac{h''}{h'} + \frac{\psi_{,z_1 z_1}}{h'}\right) z_2^2$$

$$\begin{aligned}&+\frac{h'''}{h'}z_1^2z_2+\left(h'+\frac{hh''}{h'}+\frac{2\psi_{,zz_1}}{h'}\right)z_1z_2\\&+\left(\frac{h\psi_{,z_1}}{h'}+\frac{\psi_{,z}}{h'}\right)z_2+\frac{\psi_{,zz}}{h'}z_1^2+\left(\frac{h\psi_{,z}}{h'}+\psi\right)z_1,\end{aligned}\tag{3.2}$$

where $h(z)$ and $\psi(z,z_1)$ are arbitrary differentiable functions, such that $h'\neq 0$ on a nonempty open set;

Type II:

$$\begin{aligned}z_t={}&z_4+\left[\frac{4h''}{h'}z_1+\frac{r'}{h'}\right]z_3+\frac{3h''}{h'}z_2^2+\left(\frac{6h'''}{h'}z_1^2+\frac{3r''}{h'}z_1-2h\right)z_2-\left(2h\frac{r'}{h'}+r\right)z_1\\&+\left(-2h\frac{h''}{h'}-h'\right)z_1^2+\frac{r'''}{h'}z_1^3+\frac{h^{(4)}}{h'}z_1^4,\end{aligned}\tag{3.3}$$

where $h(z)$ and $r(z)$ are arbitrary differentiable functions, such that $h'\neq 0$ on a nonempty open set.

Type III:

$$\begin{aligned}z_t={}&z_4+\left[\left(\frac{4h''}{h'}+\frac{2h'}{h}\right)z_1+\frac{\ell'}{hh'}\right]z_3+\left(\frac{3h''}{h'}+\frac{2h'}{h}\right)z_2^2\\&+\left[\frac{6h'''}{h'}+\frac{10h''}{h}-5\left(\frac{h'}{h}\right)^2\right]z_1^2z_2+\left(-\frac{3\ell'}{h^2}+\frac{3\ell''}{hh'}\right)z_1z_2+\left(-\epsilon h^2+m\right)z_2\\&+\left[\frac{h^{(4)}}{h'}+\frac{2h'''}{h}+2\frac{(h'')^2}{hh'}-\frac{5h'h''}{h^2}+2\left(\frac{h'}{h}\right)^3\right]z_1^4\\&+\left[\left(-\frac{h''}{h^2h'}+\frac{2h'}{h^3}\right)\ell'+\frac{\ell'''}{hh'}-\frac{2\ell''}{h^2}\right]z_1^3\\&+\left[\epsilon h^2\left(-\frac{h''}{h'}-2\frac{h'}{h}\right)+\left(\frac{h''}{h'}+\frac{h'}{h}\right)m\right]z_1^2\\&+\left[-\epsilon\left(\frac{h\ell'}{h'}+\ell\right)+m\frac{\ell'}{hh'}\right]z_1,\end{aligned}\tag{3.4}$$

where $h(z)$ and $\ell(z)$ are arbitrary differentiable functions, such that $h'\neq 0$ on a nonempty open set, $\epsilon=\pm 1$, and $m\neq 0$ is a real constant.

Type IV:

$$z_t=z_4+\left(\frac{h''}{h'}z_1+\frac{\phi_{,z_2}}{h'}+\lambda\right)z_3+\frac{\phi_{,z_1}}{h'}z_2+\frac{\phi_{,z}}{h'}z_1-\frac{r_0h}{h'}+\lambda\frac{\phi}{h'},\tag{3.5}$$

where $h(z)$ and $\phi(z,z_1,z_2)$ are arbitrary differentiable functions, such that $h'\neq 0$ on a nonempty open set, r_0 and λ are constants, such that $r_0^2+\lambda^2\neq 0$.

Theorem 3.1 classifies the evolution equations of the form (1.1), with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \le i \le 3$, satisfying (1.3), i.e., $f_{s1} = \mu_s f_{11} + \eta_s$, $\mu_s, \eta_s \in \mathbb{R}$, $s = 2, 3$ into four types of equations. In each case, the explicit expressions of the associated 1-forms ω_i (and hence the associated linear problems) are given as follows. Using the notation introduced in (3.1):

Type I: Equations of the form (3.2) correspond to the cases $\{\rho \neq 0, \delta_1 = 0\}$ or $\{\rho = 0, \delta_2 = 0\}$. Its associated 1-forms are described in Theorems 5.3 and 6.2 respectively.

Type II: Equations of the form (3.3) correspond to the case $\{\rho \neq 0, \delta_1 \neq 0, \alpha = 0\}$. Its associated 1-forms are described in Theorem 5.5.

Type III: Equations of the form (3.4) correspond to the cases $\{\rho \neq 0, \delta_1 \neq 0, \alpha \neq 0\}$ or $\{\rho = 0, \delta_2 \neq 0, \alpha \neq 0\}$. Its associated 1-forms are described in Theorems 5.6 and 6.5 respectively.

Type IV: Equations of the form (3.5) correspond to the case $\{\rho = 0, \delta_2 \neq 0, \alpha = 0\}$. Its associated 1-forms are described in Theorem 6.4.

We observe that equations of Type I may have distinct 1-forms and hence distinct associated linear problems, depending on the constant ρ being zero or not. A similar fact occurs with equations of Type III.

Here are some simple examples of equations described by the classification given in the paper. Further examples will be found in Section 7.

Example 3.2. Equation

$$z_t = z_4 - zz_3 - \frac{3}{2}z^3 z_1,$$

is an example of Type I. One can explicitly write the corresponding 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ where the functions f_{ij} are

$$\begin{aligned} f_{11} &= -\frac{3\epsilon z + 2\eta_2}{2\mu_2}, & f_{12} &= \frac{f_{22}}{\mu_2} + \frac{3}{2}\frac{\eta_2}{\mu_2}\left(z_2 + \frac{1}{2}zz_1 + \frac{1}{4}z^3\right), \\ f_{21} &= -\frac{3\epsilon z}{2}, & f_{22} &= \frac{3}{2\epsilon}\left(zz_2 + \frac{3}{8}z^4 - \frac{1}{2}z_1^2 - z_3\right), \\ f_{31} &= -\frac{3z + 2\epsilon\eta_2}{2\mu_2}, & f_{32} &= \epsilon\left[\frac{f_{22}}{\mu_2} + \frac{3}{2}\frac{\eta_2}{\mu_2}\left(z_2 + \frac{1}{2}zz_1 + \frac{1}{4}z^3\right)\right], \end{aligned}$$

where $\mu_2\eta_2 \neq 0$, $\epsilon = \pm 1$, $\mu_3 = \epsilon$, $\eta_3 = 0$. Observe that $\rho = 2\epsilon\eta_2\mu_2 \neq 0$, $\delta_1 = 0$. Hence the corresponding linear problem (2.5), where A and B are given by (2.4), depends on two parameters. One can also obtain a linear problem depending on three parameters.

Example 3.3. Equation

$$z_t = z_4 - zz_3 - z_1 z_2$$

is another example of Type I. One can explicitly write the corresponding 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ where the functions f_{ij} are

$$f_{11} = \frac{-\epsilon z - \eta_2}{\mu_2}, \qquad f_{12} = \frac{\epsilon}{\mu_2}\left[-z_3 + (z + \epsilon\eta_2)z_2\right],$$

$$f_{21} = -\epsilon z, \qquad f_{22} = \epsilon(-z_3 + zz_2),$$
$$f_{31} = \frac{-z - \epsilon\eta_2}{\mu_2}, \qquad f_{32} = \frac{1}{\mu_2}\big[-z_3 + (z + \epsilon\eta_2)z_2\big],$$

where $\mu_2\eta_2 \neq 0$, $\epsilon = \pm 1$, $\mu_3 = \epsilon$, $\eta_3 = 0$. One can check that $\rho = 2\epsilon\eta_2\mu_2 \neq 0$, $\delta_1 = 0$. Hence the corresponding linear problem (2.5), where A and B are given by (2.4), depends on two parameters.

The same equation, however, can also be obtained by taking $\eta_3 = \frac{\epsilon\eta_2}{\mu_3}\sqrt{\mu_3^2 - 1}$, $\mu_2 = \epsilon\frac{(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}$, $|\mu_3| > 1$, $\epsilon = \pm 1$ and 1-forms (2.2) determined by the functions f_{ij} where

$$f_{11} = \frac{\epsilon\sqrt{\mu_3^2 - 1}}{2\mu_3}\left(z - \frac{\eta_2}{\mu_3}\right), \qquad f_{12} = \frac{\epsilon\sqrt{\mu_3^2 - 1}}{2\mu_3}\left(z_3 - zz_2 + \frac{\eta_2}{\mu_3}z_2\right),$$
$$f_{21} = \frac{(1 + \mu_3^2)}{2\mu_3}z + \frac{\eta_2}{2\mu_3^2}(\mu_3^2 - 1), \qquad f_{22} = \frac{1 + \mu_3^2}{2\mu_3}(z_3 - zz_2) + \frac{(1 - \mu_3^2)}{2\mu_3^2}\eta_2 z_2,$$
$$f_{31} = \frac{\epsilon\sqrt{\mu_3^2 - 1}}{2}\left(z + \frac{\eta_2}{\mu_3}\right), \qquad f_{32} = \frac{\epsilon\sqrt{\mu_3^2 - 1}}{2}\left(z_3 - zz_2 - \frac{\eta_2}{\mu_3}z_2\right).$$

Observe that in this case $\rho = 0$ and $\delta_2 = 0$ and the 1-forms ω_i are different from the ones given above. Again the corresponding linear problem (2.5), where A and B are given by (2.4), depends on two parameters. One can also have a linear problem depending on three parameters.

Example 3.4. Equation

$$z_t = z_4 - 2zz_2 - z_1^2$$

is an example of Type II, with 1-forms (2.2) determined by the functions f_{ij}, where

$$f_{11} = \frac{-\epsilon}{\sqrt{1 + \mu_2^2}}\left(z + \frac{1}{2}\right), \qquad f_{12} = \frac{\epsilon}{\sqrt{1 + \mu_2^2}}\left(-z_3 - \mu_2 z_2 + zz_1 + \frac{z_1}{2}\right),$$
$$f_{21} = \frac{-\epsilon\mu_2}{\sqrt{1 + \mu_2^2}}\left(z + \frac{1}{2}\right), \qquad f_{22} = \frac{\epsilon}{\sqrt{1 + \mu_2^2}}\left(-\mu_2 z_3 + z_2 + \mu_2 zz_1 + \mu_2\frac{z_1}{2}\right),$$
$$f_{31} = \frac{1}{2} - z, \qquad f_{32} = -z_3 - \left(\frac{1}{2} - z\right)z_1,$$

where $\epsilon = \pm 1$. Observe that $\eta_2 = 0$, $\eta_3 = 1$, $\mu_3^2 = 1 + \mu_2^2$. Hence $\rho = -2(1 + \mu_2^2) \neq 0$, $\delta_1 = 1 + \mu_2^2 \neq 0$ and $\alpha = 0$. The corresponding linear problem (2.5), where A and B are given by (2.4), depends on the parameter μ_2.

One can also have the 1-forms (2.2) determined by the functions f_{ij}, where

$$\begin{aligned} f_{11} &= -\frac{z}{\eta_2}, & f_{12} &= -\frac{z_3}{\eta_2} + z_2 + \frac{z z_1}{\eta_2}, \\ f_{21} &= \eta_2, & f_{22} &= z_2 - \eta_2 z_1, \\ f_{31} &= \eta_2 - \frac{z}{\eta_2}, & f_{32} &= -\frac{z_3}{\eta_2} + z_2 + \frac{z z_1}{\eta_2} - \eta_2 z_1. \end{aligned}$$

Observe that $\mu_2 = 0$, $\mu_3 = 1$, $\eta_3 = \eta_2 \neq 0$. Hence, $\rho = -2\eta_2 \neq 0$, $\delta_1 = \eta_2^2 \neq 0$ and $\alpha = 0$. The corresponding linear problem (2.5), where A and B are given by (2.4), depends on the parameter η_2. One can also have a linear problem depending on three parameters.

Example 3.5. Equation

$$z_t = z_4 + 8\frac{z_1 z_3}{z} + 7\frac{z_2^2}{z} + \left(m - \epsilon z^4\right) z_2 + \left(\frac{3m}{z} - 5\epsilon z^3\right) z_1^2,$$

with $m \neq 0$, is an example of Type III. The corresponding 1-forms (2.2) are determined by the functions f_{ij} as follows

$$\begin{aligned} f_{11} &= \frac{z^2\sqrt{|\alpha|} - a}{\alpha}, \\ f_{12} &= \frac{2\epsilon z z_3}{\sqrt{|\alpha|}} + \left(\frac{14\epsilon z_1}{\sqrt{|\alpha|}} - \frac{2\epsilon\beta z}{\sqrt{|\alpha|}}\right) z_2 - \frac{6\epsilon\beta}{\sqrt{|\alpha|}} z_1^2 + \left(\frac{2\epsilon a z^3}{\alpha} - \frac{2z^5}{\sqrt{|\alpha|}}\right) z_1, \\ f_{21} &= \frac{\mu_2(h\sqrt{|\alpha|} - a)}{\alpha} + \eta_2, \\ f_{22} &= \frac{2\epsilon\mu_2 z z_3}{\sqrt{|\alpha|}} + \left(\frac{14\epsilon\mu_2 z_1}{\sqrt{|\alpha|}} - \frac{2\epsilon\eta_3 z}{\sqrt{|\alpha|}}\right) z_2 - \frac{6\epsilon\eta_3}{\sqrt{|\alpha|}} z_1^2 + \left(-\frac{2\epsilon\kappa z^3}{\alpha} - \frac{2z^5\mu_2}{\sqrt{|\alpha|}}\right) z_1, \\ f_{31} &= \frac{\mu_3(h\sqrt{|\alpha|} - a)}{\alpha} + \eta_3, \\ f_{32} &= \frac{2\epsilon\mu_3 z z_3}{\sqrt{|\alpha|}} + \left(\frac{14\epsilon\mu_3 z_1}{\sqrt{|\alpha|}} - \frac{2\epsilon\eta_2 z}{\sqrt{|\alpha|}}\right) z_2 - \frac{6\epsilon\eta_2}{\sqrt{|\alpha|}} z_1^2 + \left(\frac{2\epsilon\gamma z^3}{\alpha} - \frac{2z^5\mu_3}{\sqrt{|\alpha|}}\right) z_1, \end{aligned}$$

where a, α are given by (3.1), $\epsilon = sgn(\alpha)$, $\kappa = \mu_2\mu_3\eta_3 + (1 - \mu_3^2)\eta_2$, $\beta = \mu_3\eta_2 - \mu_2\eta_3$ and μ_2, μ_3, η_2 and η_3 are such that the constants defined in (3.1) satisfy $\rho \neq 0$, $\delta_1 \neq 0$, $\alpha \neq 0$ and $m = \delta_1/\alpha$. Hence the corresponding linear problem (2.5), where A and B are given by (2.4), in general may depend on three parameters.

Example 3.6. Equation

$$z_t = z_4 - (1 + z)z_3 - z_1 z_2$$

is an example of Type I. The corresponding 1-forms (2.2) are determined by the functions f_{ij}, where

$$
\begin{aligned}
f_{11} &= \epsilon \frac{\sqrt{\mu_3^2-1}}{2\mu_3} z, & f_{12} &= \epsilon \frac{\sqrt{\mu_3^2-1}}{2\mu_3}(z_3 - z z_2), \\
f_{21} &= \frac{1+\mu_3^2}{2\mu_3} z + \mu_3, & f_{22} &= \frac{1+\mu_3^2}{2\mu_3}(z_3 - z z_2) - \mu_3 z_2, \\
f_{31} &= \epsilon \sqrt{\mu_3^2-1}\left(\frac{z}{2}+1\right), & f_{32} &= \epsilon \sqrt{\mu_3^2-1}\left[\frac{z_3}{2} - \left(1+\frac{z}{2}\right) z_2\right],
\end{aligned}
$$

and $\epsilon = \pm 1$. Observe that $\mu_2 = \frac{\epsilon(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}$, $\eta_2 = \mu_3$ and $\eta_3 = \epsilon\sqrt{\mu_3^2-1}$. Hence $\rho = 0$ and $\delta_2 = 0$, where ρ and δ_2 are defined in (3.1). In this case, the linear problem (2.5), where A and B are given by (2.4), depends on the parameter μ_3, with $|\mu_3| > 1$.

Example 3.7. Equation

$$z_t = z_4 + z_3 + z z_1 + \frac{1}{2} z^2$$

is an example of Type IV. The corresponding 1-forms (2.2) are determined by the functions f_{ij}, where

$$
\begin{aligned}
f_{11} &= z, & f_{12} &= z_3 + \frac{z^2}{2}, \\
f_{21} &= \mu_2 z + \epsilon\sqrt{1+\mu_2^2}, & f_{22} &= \mu_2\left(\frac{z^2}{2} + z_3\right), \\
f_{31} &= \mu_2 + \epsilon z\sqrt{1+\mu_2^2}, & f_{32} &= \epsilon\left(\frac{z^2}{2} + z_3\right)\sqrt{1+\mu_2^2},
\end{aligned}
$$

and $\epsilon = \pm 1$. Observe that $\eta_2 = \mu_3 = \epsilon\sqrt{1+\mu_2^2}$ and $\eta_3 = \mu_2$. Hence $\rho = \alpha = 0$ and $\delta_2 \neq 0$, where ρ, α and δ_2 are defined in (3.1). In this case, the linear problem (2.5), where A and B are given by (2.4), depends on the parameter μ_2.

4. A characterization of differential equations which describe pseudospherical surfaces in the class $z_t = z_4 + G(z, z_1, \ldots, z_3)$

In this section, by applying the basic Theorem 2.1, we start the analysis of the system of differential equations (2.10)–(2.11) that characterizes the differential equations that describe pseudospherical surfaces of the form (1.1). In Section 4.1, we analyze this system in the general case, i.e., without any special assumption on the functions f_{ij}. Then, by assuming $f_{s1} = \mu_s f_{11} + \eta_s$, $s = 2, 3$, in Section 4.2, we present the noteworthy simplified system on which relies the rest of the paper. From now on, we will denote the derivative of a function $f(z, z_1, \ldots, z_r)$ with respect to z_i by $f_{,z_i} = \frac{\partial f}{\partial z_i}$, $i \geq 1$, $f_{,z} = \frac{\partial f}{\partial z}$. Whenever a function depends only on z, we denote its derivatives by f', f'', etc.

4.1. The general case

As a consequence of Theorem 2.1, the following theorem gives a characterization of the differential equations which describe pseudospherical surfaces of the form (1.1).

Theorem 4.1. *An evolution equation*

$$z_t = z_4 + G(z, z_1, z_2, z_3),$$

describes pseudospherical surfaces if, and only if, the functions f_{ij} *have the form*

$$f_{i1} = f_{i1}(z), \qquad f_{i2} = f_{i2}(z, ..., z_3), \tag{4.1}$$

where

$$\begin{cases} f_{12} = z_3 f'_{11} + \phi_{12}(z, z_1, z_2), \\ f_{22} = z_3 f'_{21} + \phi_{22}(z, z_1, z_2), \\ f_{32} = z_3 f'_{31} + \phi_{32}(z, z_1, z_2), \end{cases} \tag{4.2}$$

the function G *has the form*

$$G = g_1(z, z_1, z_2) z_3 + g_2(z, z_1, z_2), \tag{4.3}$$

and

$$\begin{aligned} g_1 = \frac{1}{\Delta}\big[&\big(f'_{21} f''_{21} + f'_{11} f''_{11} + f'_{31} f''_{31}\big) z_1 + f'_{11}\phi_{12,z_2} + 2 f'_{31}\big(f_{21} f'_{11} - f_{11} f'_{21}\big) + f'_{31}\phi_{32,z_2} \\ &+ f'_{21}\phi_{22,z_2}\big], \end{aligned} \tag{4.4}$$

$$\begin{aligned} g_2 = \frac{1}{\Delta}\big[&\big(f'_{21} f_{31} + f_{21} f'_{31}\big)\phi_{12} - \big(f'_{11} f_{31} + f_{11} f'_{31}\big)\phi_{22} + \big(f'_{11} f_{21} - f_{11} f'_{21}\big)\phi_{32} \\ &+ f'_{11}(\phi_{12,z} z_1 + \phi_{12,z_1} z_2) + f'_{21}(\phi_{22,z} z_1 + \phi_{22,z_1} z_2) + f'_{31}(\phi_{32,z} z_1 + \phi_{32,z_1} z_2)\big], \end{aligned} \tag{4.5}$$

Δ *is given by (2.9) and* ϕ_{12}, ϕ_{22}, ϕ_{32} *are differentiable functions which satisfy*

$$\phi_{22,z_2} - g_1 f'_{21} + z_1 f''_{21} + f_{31} f'_{11} - f_{11} f'_{31} = 0, \tag{4.6}$$

$$\phi_{32,z_2} - g_1 f'_{31} + z_1 f''_{31} + f_{21} f'_{11} - f_{11} f'_{21} = 0, \tag{4.7}$$

$$f_{31}\phi_{12} - f_{11}\phi_{32} - g_2 f'_{21} + z_2\phi_{22,z_1} + z_1\phi_{22,z} = 0, \tag{4.8}$$

$$f_{21}\phi_{12} - f_{11}\phi_{22} - g_2 f'_{31} + z_2\phi_{32,z_1} + z_1\phi_{32,z} = 0. \tag{4.9}$$

Proof. In view of Theorem 2.1, an evolution equation of the form (1.1) describes pseudospherical surfaces if, and only if, the functions f_{ij} satisfy (2.8)–(2.10) and the differential equations (2.11). Hence f_{ij} have the form (4.1) and the condition $F = z_4 + G$ entails that equations (2.11) are linear in z_4 and reduce to

$$\begin{cases} (-f'_{11}+f_{12,z_3})z_4+f_{12,z_2}z_3+f_{12,z_1}z_2+f_{12,z}z_1+f_{21}f_{32}-f_{22}f_{31}-Gf'_{11}=0,\\ (-f'_{21}+f_{22,z_3})z_4+f_{22,z_2}z_3+f_{22,z_1}z_2+f_{22,z}z_1+f_{12}f_{31}-f_{11}f_{32}-Gf'_{21}=0,\\ (-f'_{31}+f_{32,z_3})z_4+f_{32,z_2}z_3+f_{32,z_1}z_2+f_{32,z}z_1+f_{12}f_{21}-f_{11}f_{22}-Gf'_{31}=0. \end{cases} \tag{4.10}$$

Hence this system splits into 6 equations, among which the three simple ones, that are obtained from the coefficients of z_4,

$$-f'_{11}+f_{12,z_3}=0,\qquad -f'_{21}+f_{22,z_3}=0,\qquad -f'_{31}+f_{32,z_3}=0,$$

can be integrated in z_3, using (4.1). Therefore, f_{12}, f_{22} and f_{32} are given by (4.2).

Substituting (4.2) into the system (4.10) and taking the derivative twice with respect to z_3, one gets

$$G_{z_3z_3}f'_{11}=0,\qquad G_{z_3z_3}f'_{21}=0,\qquad G_{z_3z_3}f'_{31}=0,$$

where f'_{11}, f'_{21} and f'_{31} cannot be all zero, due to condition (2.9). Hence G has the form given by (4.3) and the system (4.10) can be written in the form

$$\begin{cases} A_1z_3+A_2=0,\\ B_1z_3+B_2=0,\\ C_1z_3+C_2=0, \end{cases}$$

with coefficients

$$\begin{aligned} A_1&=\phi_{12,z_2}-g_1f'_{11}+z_1f''_{11}+f_{21}f'_{31}-f_{31}f'_{21},\\ B_1&=\phi_{22,z_2}-g_1f'_{21}+z_1f''_{21}+f_{31}f'_{11}-f_{11}f'_{31},\\ C_1&=\phi_{32,z_2}-g_1f'_{31}+z_1f''_{31}+f_{21}f'_{11}-f_{11}f'_{21}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} A_2&=f_{21}\phi_{32}-f_{31}\phi_{22}-g_2f'_{11}+z_2\phi_{12,z_1}+z_1\phi_{12,z},\\ B_2&=f_{31}\phi_{12}-f_{11}\phi_{32}-g_2f'_{21}+z_2\phi_{22,z_1}+z_1\phi_{22,z},\\ C_2&=f_{21}\phi_{12}-f_{11}\phi_{22}-g_2f'_{31}+z_2\phi_{32,z_1}+z_1\phi_{32,z} \end{aligned} \tag{4.12}$$

not depending on z_3. Therefore (4.10) is equivalent to six equations $\{A_j=0,\ B_j=0,\ C_j=0,\ j=1,2\}$.

By using these equations, one can readily get explicit formulas for the functions g_1 and g_2. Indeed, in view of (2.9), the following two linear combinations

$$f'_{11}A_1+f'_{21}B_1+f'_{31}C_1=0$$

and

$$f'_{11}A_2 + f'_{21}B_2 + f'_{31}C_2 = 0$$

can be solved with respect to g_1 and g_2. In fact, we get g_1 and g_2 given by (4.4) and (4.5). It follows that, modulo these two expressions for g_1 and g_2, if $f'_{11} \neq 0$ or $f'_{21} \neq 0$ or $f'_{31} \neq 0$, then the system (4.10) reduces to the four equations $B_i = C_i = 0$, with $i = 1, 2$, which are precisely equations (4.6)–(4.9). □

The analysis of equations (4.6)–(4.9) in general is quite complicated and in the following subsection we will make an assumption which noteworthy simplifies the problem.

4.2. *Assuming* $f_{s1} = \mu_s f_{11} + \eta_s$, $s = 2, 3$

The following theorem gives a characterization of equations which describe pseudospherical surfaces of the form (1.1), assuming that (1.3) holds, i.e.

$$f_{s1} = \mu_s f_{11} + \eta_s, \qquad s = 2, 3.$$

Theorem 4.2. *An evolution equation of the form (1.1) describes pseudospherical surfaces with the assumption (1.3) if, and only if,* f_{ij} *satisfy (4.1) and (4.2),* G *satisfies (4.3) where*

$$g_1 = \frac{1}{f'_{11}(1+\mu_2^2+\mu_3^2)}\Big[\big(1+\mu_2^2+\mu_3^2\big) f''_{11} z_1 + 2\mu_3\eta_2 f'_{11} + \phi_{12,z_2} + \mu_2\phi_{22,z_2} + \mu_3\phi_{32,z_2}\Big], \tag{4.13}$$

$$\begin{aligned} g_2 = \frac{1}{f'_{11}(1+\mu_2^2+\mu_3^2)}\Big[&(2\mu_2\mu_3 f_{11} + \mu_3\eta_2 + \mu_2\eta_3)\phi_{12} + \phi_{12,z} z_1 + \phi_{12,z_1} z_2 \\ &- (\eta_3 + 2\mu_3 f_{11})\phi_{22} + \mu_2(\phi_{22,z} z_1 + \phi_{22,z_1} z_2) + \eta_2\phi_{32} + \mu_3(\phi_{32,z} z_1 + \phi_{32,z_1} z_2)\Big], \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \phi_{22} &= (\mu_3 a - \eta_3\alpha) f'_{11} z_2 + \mu_2\phi_{12} + \psi_{22}, \\ \phi_{32} &= (\mu_2 a - \eta_2\alpha) f'_{11} z_2 + \mu_3\phi_{12} + \psi_{32}, \end{aligned} \tag{4.15}$$

where a, α *are given by (3.1) and* $\phi_{12} = \phi_{12}(z, z_1, z_2)$, $\psi_{22} = \psi_{22}(z, z_1)$, $\psi_{32} = \psi_{32}(z, z_1)$ *are functions which satisfy*

$$\begin{aligned} -\rho\phi_{12} + \big\{&\rho z_1 f''_{11} + \big[\big(\big(1+\mu_3^2\big)\eta_2 - \mu_2\mu_3\eta_3\big)\alpha f_{11} - \mu_2\delta_1\big] f'_{11} \\ &+ \big(1+\mu_3^2\big)\psi_{22,z_1} - \mu_2\mu_3\psi_{32,z_1}\big\} z_2 + \mu_2(2\mu_3 f_{11} + \eta_3)\psi_{22} - \mu_2\mu_3 z_1\psi_{32,z} \\ &+ z_1\big(1+\mu_3^2\big)\psi_{22,z} - \big[\eta_2\mu_2 + \big(1+\mu_2^2+\mu_3^2\big) f_{11}\big]\psi_{32} = 0, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&\bigl\{-\eta_2\alpha z_1 f''_{11} + \bigl(1+\mu_2^2\bigr)\psi_{32,z_1} + \bigl[(\alpha\eta_3 - a\mu_3)\alpha f_{11} - \mu_3\delta_1\bigr] f'_{11} - \mu_2\mu_3\psi_{22,z_1}\bigr\} z_2\\ &\quad + \bigl[\bigl(1+\mu_2^2\bigr)\psi_{32,z} - \mu_2\mu_3\psi_{22,z}\bigr] z_1 + (-\alpha f_{11} + \mu_3\eta_3)\psi_{22} - \mu_3\eta_2\psi_{32} + \eta_2\alpha\phi_{12} = 0.\end{aligned} \tag{4.17}$$

Proof. The assumption (1.3) and the nondegeneracy condition (2.9) entail that $f'_{11} \neq 0$. Then, using (4.4) and (4.5) one gets (4.13) and (4.14). Now equations (4.6)–(4.7) can be integrated with respect to z_2. In fact, these equations are respectively

$$\phi_{22,z_2} = (\mu_3 a - \eta_3\alpha) f'_{11} + \mu_2\phi_{12,z_2}, \qquad \phi_{32,z_2} = (\mu_2 a - \eta_2\alpha) f'_{11} + \mu_3\phi_{12,z_2}.$$

Hence we obtain (4.15). Now, substituting (4.15), together with $f_{s1} = \mu_s f_{11} + \eta_s$, into (4.8) and (4.9), one can see that these equations reduce to (4.16) and (4.17), respectively. □

Equations (4.16)–(4.17) can be completely solved by analyzing all possible cases occurring when $\rho = 0$ and $\rho \neq 0$. The diagrams of Figs. 3.1 and 3.2 give a summary of the analysis that will be given in Sections 5 and 6.

5. Case $\rho \neq 0$

This section is devoted to the characterization of equations that describe pseudospherical surfaces of the form (1.1), whose associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \le i \le 3$, are such that f_{ij} satisfy (1.3) and $\rho \neq 0$. The analysis of this case naturally splits into three further cases. In Section 5.1, we consider the first case $\{\rho \neq 0, \delta_1 = 0\}$, whereas in Section 5.2 we treat the other two cases, which are $\{\rho \neq 0, \delta_1 \neq 0, \alpha = 0\}$ and $\{\rho \neq 0, \delta_1 \neq 0, \alpha \neq 0\}$, respectively. We start with some preliminaries.

Lemma 5.1. *Under the assumptions of Theorem 4.2, if $\rho \neq 0$, then (4.16) is equivalent to*

$$\begin{aligned}\phi_{12}(z, z_1, z_2) = \frac{1}{\rho}\bigl\{&\rho z_1 f''_{11} + \bigl[\bigl(\bigl(1+\mu_3^2\bigr)\eta_2 - \mu_2\mu_3\eta_3\bigr)\alpha f_{11} - \mu_2\delta_1\bigr] f'_{11}\\ &+ \bigl(1+\mu_3^2\bigr)\psi_{22,z_1} - \mu_2\mu_3\psi_{32,z_1}\bigr\} z_2 + \mu_2(2\mu_3 f_{11} + \eta_3)\psi_{22}\\ &- \bigl[\eta_2\mu_2 + \bigl(1+\mu_2^2+\mu_3^2\bigr) f_{11}\bigr]\psi_{32} + z_1\bigl(1+\mu_3^2\bigr)\psi_{22,z}\\ &- \mu_2\mu_3 z_1\psi_{32,z} = 0,\end{aligned} \tag{5.1}$$

and (4.17) is equivalent to the following two equations

$$(-\mu_2 a + \eta_2\alpha)\psi_{22} + (\mu_3 a - \eta_3\alpha)\psi_{32} = \delta_1 f'_{11} z_1 L + r_0, \tag{5.2}$$

$$\eta_3\psi_{22} - \eta_2\psi_{32} = -\frac{z_1\partial_z(\delta_1 f'_{11} z_1 L + r_0)}{L}, \tag{5.3}$$

where $r_0 = r_0(z)$ is an arbitrary function, $\alpha^2 + a^2 \neq 0$ and

$$L = \alpha f_{11} + a \tag{5.4}$$

is nonzero on a nonempty open set.

Proof. If $\rho \neq 0$, then (4.16) is clearly equivalent to (5.1). Then, equation (4.17) simply reduces to

$$C_{21} z_2 + C_{20} = 0,$$

where

$$C_{21} := (-\mu_2 a + \eta_2 \alpha)\psi_{22,z_1} + (\mu_3 a - \eta_3 \alpha)\psi_{32,z_1} - \delta_1 f'_{11}(\alpha f_{11} + a),$$
$$C_{20} := z_1\big[(-\mu_2 a + \eta_2 \alpha)\psi_{22,z} + (\mu_3 a - \eta_3 \alpha)\psi_{32,z}\big] + (\alpha f_{11} + a)(\eta_3 \psi_{22} - \eta_2 \psi_{32}),$$

do not depend on z_2. Hence, $C_2 = 0$ is equivalent to $C_{21} = C_{20} = 0$, i.e., to the system

$$\begin{cases} (-\mu_2 a + \eta_2 \alpha)\psi_{22,z_1} + (\mu_3 a - \eta_3 \alpha)\psi_{32,z_1} - \delta_1 f'_{11} L = 0, \\ z_1\big[(-\mu_2 a + \eta_2 \alpha)\psi_{22,z} + (\mu_3 a - \eta_3 \alpha)\psi_{32,z}\big] + (\eta_3 \psi_{22} - \eta_2 \psi_{32}) L = 0, \end{cases} \tag{5.5}$$

where L, given by (5.4), is nonzero on a nonempty open set. In fact, since $f'_{11} \neq 0$, $L = 0$ entails $\alpha = a = 0$ which is not compatible with $\rho \neq 0$.

Due to (4.1), these two equations can be easily integrated in z_1. In fact, from the first equation of (5.5) we get (5.2) and substituting into the second equation of (5.5), one gets

$$(\eta_3 \psi_{22} - \eta_2 \psi_{32}) L = -\delta_1 \partial_z \big(f'_{11} z_1 L + r_0\big) z_1. \tag{5.6}$$

Hence, (5.6) can be rewritten as (5.3). □

Now, equations (5.2) and (5.3) form a linear system for ψ_{22}, ψ_{32} whose determinant is equal to δ_1. For the analysis of this system, it is useful to use the following

Lemma 5.2. *Let ρ, α, a and δ_1 be the constants defined by (3.1). If $\rho \neq 0$ and $\delta_1 = 0$, then $\mu_3 a - \eta_3 \alpha \neq 0$ and $\eta_2 \neq 0$.*

Proof. Assume $\eta_2 = 0$. Since $\delta_1 = \eta_2(\mu_2 a - \eta_2 \alpha) - \eta_3(\mu_3 a - \eta_3 \alpha) = 0$, it follows that $\eta_3(\mu_3 a - \eta_3 \alpha) = 0$. If $\eta_3 = 0$ then $\rho = 0$ which is a contradiction. Hence $\eta_3 \neq 0$ and $\mu_3 a - \eta_3 \alpha = 0$. Since $a = -\mu_3 \eta_3$, we get $\alpha = -\mu_3^2$ and hence $1 + \mu_2^2 = 0$, which is a contradiction. Therefore, we conclude that $\eta_2 \neq 0$.

In order to prove that $\mu_3 a - \eta_3 \alpha \neq 0$, assume otherwise that $\mu_3 a - \eta_3 \alpha = 0$, then since $\eta_2 \neq 0$ it follows from the fact that $\delta_1 = 0$ that $\mu_2 a - \eta_2 \alpha = 0$. Observe that $\rho \neq 0$ implies that $a \neq 0$. Therefore, the determinant of the linear system for α and a vanishes, i.e., $\mu_2 \eta_3 - \mu_3 \eta_2 = 0$. Since $\delta_1 = (\mu_2 \eta_3 - \mu_3 \eta_2)^2 - \eta_2^2 + \eta_3^2 = 0$ it follows that $\eta_3 = \pm \eta_2 \neq 0$ and therefore $\mu_3 = \pm \mu_2$. This contradicts the fact that $a \neq 0$. □

The complete and explicit description of the solutions ψ_{22}, ψ_{32} of (5.2) and (5.3) are given in Section 5.1 when $\{\rho \neq 0, \delta_1 = 0\}$ and Section 5.2 when $\{\rho \neq 0, \delta_1 \neq 0\}$.

5.1. Case $\{\rho \neq 0, \delta_1 = 0\}$

Theorem 5.3. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$ *satisfying (1.3), with* $\{\rho \neq 0, \delta_1 = 0\}$ *(*ρ *and* δ_1 *are the constants defined by (3.1)) if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = z_4 &+ \left[\frac{2h''}{h'}z_1 + h + \frac{\psi_{,z_1}}{h'}\right]z_3 + \left(\frac{h''}{h'} + \frac{\psi_{,z_1z_1}}{h'}\right)z_2^2 \\ &+ \frac{h'''}{h'}z_1^2z_2 + \left(h' + \frac{hh''}{h'} + \frac{2\psi_{,zz_1}}{h'}\right)z_1z_2 + \left(\frac{h\psi_{,z_1}}{h'} + \frac{\psi_{,z}}{h'}\right)z_2 \\ &+ \frac{\psi_{,zz}}{h'}z_1^2 + \left(\frac{h\psi_{,z}}{h'} + \psi\right)z_1, \end{aligned} \tag{5.7}$$

where $h(z)$ *and* $\psi(z, z_1)$ *are arbitrary differentiable functions, such that* $h' \neq 0$ *on a nonempty open set, and the functions* f_{ij} *are given by*

$$f_{11} = \frac{\eta_2}{\gamma}\left(h - \epsilon\sqrt{\eta_2^2 - \eta_3^2}\right), \qquad f_{21} = \mu_2 f_{11} + \eta_2, \qquad f_{31} = \mu_3 f_{11} + \eta_3, \tag{5.8}$$

$$f_{12} = f_{11}'z_3 + \left(f_{11}''z_1 + \frac{\gamma f_{11}f_{11}'}{\eta_2} + \eta_2\frac{\psi_{,z_1}}{\gamma}\right)z_2 + \eta_2\frac{\psi_{,z}}{\gamma}z_1 + f_{11}\psi, \tag{5.9}$$

$$\begin{aligned} f_{22} = \mu_2 f_{11}'z_3 &+ \left[\mu_2 f_{11}''z_1 + \left(\frac{\mu_2\gamma f_{11}}{\eta_2} + \gamma\right)f_{11}' + \frac{\mu_2\eta_2\psi_{,z_1}}{\gamma}\right]z_2 \\ &+ \frac{\mu_2\eta_2\psi_{,z}}{\gamma}z_1 + \eta_2\left(\frac{\mu_2 f_{11}}{\eta_2} + 1\right)\psi, \end{aligned} \tag{5.10}$$

$$\begin{aligned} f_{32} = \mu_3 f_{11}'z_3 &+ \left[\mu_3 f_{11}''z_1 + \frac{\gamma(\mu_3 f_{11} + \eta_3)}{\eta_2}f_{11}' + \frac{(\eta_3\gamma + 2\eta_2^2)\mu_3}{\rho\eta_2}\psi_{,z_1}\right]z_2 \\ &+ \frac{(\eta_3\gamma + 2\eta_2^2)\mu_3\psi_{,z}}{\rho\eta_2}z_1 + (\mu_3 f_{11} + \eta_3)\psi, \end{aligned} \tag{5.11}$$

where $\eta_2\gamma \neq 0$ *and*

$$\mu_3 = \frac{\mu_2\eta_3 + \epsilon\sqrt{\eta_2^2 - \eta_3^2}}{\eta_2}, \qquad \gamma = \epsilon\mu_2\sqrt{\eta_2^2 - \eta_3^2} - \eta_3, \qquad \epsilon := \pm 1. \tag{5.12}$$

Proof. Since $\rho \neq 0$ and $\delta_1 = 0$, Lemma 5.2 entails that $\eta_2 \neq 0$ and $\mu_3 a - \eta_3\alpha \neq 0$. Hence (5.2) can be solved in the form

$$\psi_{32} = \frac{r_0 + (\mu_2 a - \eta_2\alpha)\psi_{22}}{\mu_3 a - \eta_3\alpha}, \tag{5.13}$$

whereas (5.3) gives

$$\psi_{32} = \frac{r_0'z_1}{\eta_2 L} + \frac{\eta_3}{\eta_2}\psi_{22}, \tag{5.14}$$

where $L=\alpha f_{11}+a$. Since $\delta_1=0$, comparing (5.13) and (5.14), one gets

$$\frac{r_0' z_1}{\eta_2 L}=\frac{r_0}{\mu_3 a-\eta_3\alpha}.$$

It follows that $r_0=0$, and

$$\psi_{32}=\frac{\eta_3}{\eta_2}\psi_{22}. \tag{5.15}$$

Now, since $\delta_1=0$, one can consider the following identities

$$\gamma(\mu_3\beta+\eta_2)=\rho\eta_2, \qquad \frac{(\mu_2 a-\eta_2\alpha)}{\mu_3 a-\eta_3\alpha}=\frac{\eta_3}{\eta_2}. \tag{5.16}$$

where $\beta=\mu_3\eta_2-\eta_3\mu_2$. Substituting (5.14) and (5.15) into (5.1), (5.13), (4.2), (4.15) and (4.13)–(4.14), one gets that the general form of an equation which describes pseudospherical surfaces is

$$\begin{aligned} z_t={}&z_4+\left[\frac{2f_{11}''}{f_{11}'}z_1+\frac{\psi_{22,z_1}}{\gamma f_{11}'}+\frac{\gamma}{\eta_2}f_{11}+\beta\right]z_3+\left(\frac{f_{11}''}{f_{11}'}+\frac{\psi_{22,z_1z_1}}{\gamma f_{11}'}\right)z_2^2+\frac{f_{11}'''}{f_{11}'}z_1^2z_2\\ &+\left[\left(\frac{\gamma f_{11}}{\eta_2}+\beta\right)\frac{f_{11}''}{f_{11}'}+\frac{\gamma f_{11}'}{\eta_2}+\frac{2\psi_{22,zz_1}}{\gamma f_{11}'}\right]z_1z_2+\left[\left(\frac{\gamma f_{11}}{\eta_2}+\beta\right)\frac{\psi_{22,z_1}}{\gamma f_{11}'}+\frac{\psi_{22,z}}{\gamma f_{11}'}\right]z_2\\ &+\frac{\psi_{22,zz}}{\gamma f_{11}'}z_1^2+\left[\frac{\psi_{22}}{\eta_2}+\left(\frac{\gamma f_{11}}{\eta_2}+\beta\right)\frac{\psi_{22,z}}{\gamma f_{11}'}\right]z_1. \end{aligned} \tag{5.17}$$

Notice that $\gamma\neq 0$, since otherwise (5.16) would imply $\rho=0$. Moreover, $\delta_1=0$ implies that μ_3 and γ are given by (5.12). Then, defining new functions $h=\frac{\gamma f_{11}}{\eta_2}+\beta$ and $\psi=\frac{\psi_{22}}{\eta_2}$, formula (5.17) reduces to (5.7) and the functions f_{ij} are given by (5.8)–(5.11). Now, the functions f_{ij} satisfy the generic condition (2.8). In fact, the coefficient of z_3 in $f_{11}f_{22}-f_{12}f_{21}$ is $-\eta_2 f_{11}'$, which is nonzero since $\eta_2\neq 0$ and $f_{11}'\neq 0$.

The converse of Theorem 5.3 is a straightforward computation. □

Observe that equation (5.7) coincides with (3.2) and it is called of Type I in our main classification result, Theorem 3.1. It follows from Theorem 5.3 that by choosing $h(z)$ and $\psi(z,z_1)$ to be arbitrary differentiable functions, independent of any parameter, such that h' is nonzero, the differential equation (5.7) is the integrability condition of a linear problem (2.5), determined by functions f_{ij} which are given in (5.8)–(5.11) in terms of h, ψ and the parameters μ_2,η_2,η_3.

An example of such an equation is given by Example 3.2, which is obtained by choosing in Theorem 5.3 $h=-\frac{3}{2}z$, $\psi=-\frac{3}{4}zz_1-\frac{3}{8}z^3$, and $\mu_2\neq 0$, $\epsilon=\pm 1$, $\mu_3=\epsilon$, $\eta_3=0$. See also Example 3.3, Examples 7.1 and 7.2.

5.2. *Case* $\{\rho\neq 0,\delta_1\neq 0\}$

This section is devoted to the complete description of the differential equations that describe pseudospherical surfaces of the form (1.1), with associated 1-forms $\omega_i=f_{i1}dx+f_{i2}dt$, satisfying (1.3) and the conditions $\rho\neq 0$, $\delta_1\neq 0$. One has the following preliminary lemma.

Lemma 5.4. *If $\rho \neq 0$ and $\delta_1 \neq 0$, then (5.2)–(5.3) are equivalent to*

$$\begin{aligned}\psi_{22} = {} & z_1^2\left(\mu_2\mu_3\eta_2 - \eta_3 - \eta_3\mu_2^2\right)\left[f_{11}'' + \frac{\alpha (f_{11}')^2}{\alpha f_{11} + a}\right] \\ & + \left\{\frac{(\mu_2\mu_3\eta_2 - \eta_3 - \eta_3\mu_2^2)r'}{\alpha f_{11} + a} - (\alpha f_{11} + a)\eta_2 f_{11}'\right\} z_1 - \eta_2 r, \qquad (5.18)\\ \psi_{32} = {} & z_1^2\left(-\mu_2\mu_3\eta_3 - \eta_2 + \eta_2\mu_3^2\right)\left[f_{11}'' + \frac{\alpha (f_{11}')^2}{\alpha f_{11} + a}\right] \\ & + z_1\left\{\frac{(-\mu_2\mu_3\eta_3 - \eta_2 + \eta_2\mu_3^2)r'}{\alpha f_{11} + a} - [\alpha f_{11} + a]\eta_3 f_{11}'\right\} z_1 - \eta_3 r, \qquad (5.19)\end{aligned}$$

where $r = r(z)$ is an arbitrary differentiable function.

Proof. Since $\delta_1 \neq 0$, (5.2) and (5.3) give a nondegenerate linear system with respect to ψ_{22}, ψ_{32}. Hence, one obtains (5.18) and (5.19) by solving these equations with respect to ψ_{22}, ψ_{32}. □

In the remaining of this section, it will be convenient to distinguish between the special case when $\alpha = 0$ given in Theorem 5.5 and the generic case $\alpha \neq 0$ given in Theorem 5.6.

Theorem 5.5. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ satisfying (1.3) and $\{\rho \neq 0, \delta_1 \neq 0, \alpha = 0\}$ (ρ, δ_1 and α are the constants defined by (3.1)) if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = {} & z_4 + \left[\frac{4h''}{h'} z_1 + \frac{r'}{h'}\right] z_3 + \frac{3h''}{h'} z_2^2 + \left(\frac{6h'''}{h'} z_1^2 + \frac{3r''}{h'} z_1 - 2h\right) z_2 \\ & - \left(2h\frac{h''}{h'} + h'\right) z_1^2 - \left(2h\frac{r'}{h'} + r\right) z_1 + \frac{h^{(4)}}{h'} z_1^4 + \frac{r'''}{h'} z_1^3, \qquad (5.20)\end{aligned}$$

where $h(z)$ and $r(z)$ are arbitrary differentiable functions, such that $h' \neq 0$ on a nonempty open set, and the functions f_{ij} are given by

$$f_{11} = \frac{1}{a}\left(h - \frac{\eta_2^2 - \eta_3^2,}{2}\right), \qquad f_{21} = \mu_2 f_{11} + \eta_2, \qquad f_{31} = \epsilon\sqrt{1 + \mu_2^2}\, f_{11} + \eta_3, \qquad (5.21)$$

$$\begin{aligned} f_{12} = {} & f_{11}' z_3 + \left[3 f_{11}'' z_1 - \frac{(\eta_2 + a\mu_2) f_{11}'}{\epsilon\sqrt{1 + \mu_2^2}} + \frac{r'}{a}\right] z_2 + f_{11}''' z_1^3 + \left[\frac{r''}{a} - \frac{(a\mu_2 + \eta_2)}{\epsilon\sqrt{1 + \mu_2^2}} f_{11}''\right] z_1^2 \\ & - \left[a f_{11} f_{11}' + \frac{(\eta_2 + a\mu_2) r'}{\epsilon a\sqrt{1 + \mu_2^2}}\right] z_1 - f_{11} r, \qquad (5.22)\end{aligned}$$

$$\begin{aligned} f_{22} = {} & \mu_2 f_{11}' z_3 + \left[3 f_{11}'' \mu_2 z_1 - f_{11}' \eta_3 + \frac{r'\mu_2}{a}\right] z_2 - \left[(\mu_2 f_{11} + \eta_2) a f_{11}' + \frac{r'\eta_3}{a}\right] z_1 \\ & + \left[\frac{\mu_2 r''}{a} - \eta_3 f_{11}''\right] z_1^2 + \mu_2 f_{11}''' z_1^3 - r\mu_2 f_{11} - \eta_2 r, \qquad (5.23)\end{aligned}$$

$$\begin{aligned} f_{32} &= \epsilon\sqrt{1+\mu_2^2}f'_{11}z_3 + \left[\epsilon\sqrt{1+\mu_2^2}\left(3f''_{11}z_1 + \frac{r'}{a}\right) - f'_{11}\eta_2\right]z_2 + \epsilon\sqrt{1+\mu_2^2}f'''_{11}z_1^3 \\ &\quad - \left[\left(\epsilon a f_{11}\sqrt{1+\mu_2^2} + a\eta_3\right)f'_{11} + \frac{r'\eta_2}{a}\right]z_1 + \left(\frac{\epsilon r''\sqrt{1+\mu_2^2}}{a} - \eta_2 f''_{11}\right)z_1^2 \\ &\quad - \epsilon r f_{11}\sqrt{1+\mu_2^2} - \eta_3 r, \end{aligned} \tag{5.24}$$

where $\epsilon = \pm 1$ *and*

$$a := \mu_2\eta_2 - \epsilon\eta_3\sqrt{1+\mu_2^2} \neq 0. \tag{5.25}$$

Proof. By hypothesis $\alpha = 0$, hence it follows from (3.1) that $\rho = 2\mu_3 a \neq 0$, $\mu_3^2 = 1 + \mu_2^2$, i.e., $\mu_3 = \epsilon\sqrt{1+\mu_2^2}$, where $\epsilon = \pm 1$ and hence a is given by (5.25). Taking into account (5.1), (5.18) and (5.19), equations (4.2), (4.15) and (4.13)–(4.14) lead to the functions f_{ij} given by (5.21)–(5.24) and to the following general form for the differential equations that describe pseudospherical surfaces

$$\begin{aligned} z_t &= z_4 + \left[\frac{4f''_{11}}{f'_{11}}z_1 + \frac{r'}{af'_{11}}\right]z_3 + \frac{3f''_{11}}{f'_{11}}z_2^2 + \left(\frac{6f'''_{11}}{f'_{11}}z_1^2 + \frac{3r''}{af'_{11}}z_1 - 2af_{11} - b\right)z_2 \\ &\quad - \left[\left(2f_{11} + \frac{b}{a}\right)\frac{r'}{f'_{11}} - r\right]z_1 - \left[(2af_{11} + b)\frac{f''_{11}}{f'_{11}} + af'_{11}\right]z_1^2 + \frac{f^{(4)}_{11}}{f'_{11}}z_1^4 + \frac{r'''}{af'_{11}}z_1^3, \end{aligned} \tag{5.26}$$

where $b = \eta_2^2 - \eta_3^2$. Then by defining the function $h = af_{11} + b/2$, we have that (5.21) holds. Now (5.20) follows by substituting f_{11} given by (5.21) into (5.26).

Finally, we have to verify that the functions f_{ij} satisfy the generic condition (2.8). If $\eta_2 \neq 0$, then the coefficient of z_3 in $f_{11}f_{22} - f_{12}f_{21}$ is nonzero on an open set. If $\eta_2 = 0$ then, the coefficient of z_2 reduces to $(1+\mu_2^2)f_{11}f'_{11}$ which is nonzero on an open set. Therefore, (2.8) is satisfied.

The converse of Theorem 5.5 is a straightforward computation. □

Observe that equation (5.20) coincides with (3.3) and it is called of Type II in our main classification result, Theorem 3.1. In view of Theorem 5.5 one can see that, when $h(z)$ and $r(z)$ are arbitrary differentiable functions independent of any parameter such that $h' \neq 0$, then (5.20) is a differential equation that describes pseudospherical surfaces which is the integrability condition of a linear problem that depends on the three parameters μ_2, η_2 and η_3.

An example of such an equation is given by Example 3.4, which is obtained by choosing in Theorem 5.5 $h = -z$, $r = 0$, $\epsilon = 1$, $\eta_3 = \eta_2 \neq 0$, $\mu_2 = 0$. See also Example 7.3.

Our next theorem considers the generic case where $\rho \neq 0$, $\delta_1 \neq 0$ and $\alpha \neq 0$. As a consequence of this fact the corresponding evolution equations are more complicated.

Theorem 5.6. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$ *satisfying (1.3) and* $\{\rho \neq 0, \delta_1 \neq 0, \alpha \neq 0\}$ *(*ρ*,* δ_1 *and* α *are the constants defined by (3.1)) if, and only if, the differential equation has the form*

$$
\begin{aligned}
z_t = {} & z_4 + \left[\left(\frac{4h''}{h'} + \frac{2h'}{h}\right)z_1 + \frac{\ell'}{hh'}\right]z_3 + \left(\frac{3h''}{h'} + \frac{2h'}{h}\right)z_2^2 \\
& + \left\{\left[\frac{6h'''}{h'} + \frac{10h''}{h} - 5\left(\frac{h'}{h}\right)^2\right]z_1^2 + 3\left(-\frac{\ell'}{h^2} + \frac{\ell''}{hh'}\right)z_1 - \epsilon h^2 + m\right\}z_2 \\
& + \left[\frac{h^{(4)}}{h'} + \frac{2h'''}{h} + 2\frac{(h'')^2}{hh'} - \frac{5h'h''}{h^2} + 2\left(\frac{h'}{h}\right)^3\right]z_1^4 \\
& + \left[\left(-\frac{h''}{h^2h'} + \frac{2h'}{h^3}\right)\ell' + \frac{\ell'''}{hh'} - \frac{2\ell''}{h^2}\right]z_1^3 + \left[-\epsilon h^2\left(\frac{h''}{h'} + 2\frac{h'}{h}\right) + m\left(\frac{h''}{h'} + \frac{h'}{h}\right)\right]z_1^2 \\
& + \left[-\epsilon\left(h^2\frac{\ell'}{hh'} + \ell\right) + m\frac{\ell'}{hh'}\right]z_1, \qquad (5.27)
\end{aligned}
$$

where $\epsilon = \pm 1$, $m \in \mathbb{R}\setminus\{0\}$, $h(z)$ and $\ell(z)$ are arbitrary differentiable functions, such that h' does not vanish on a nonempty open set, and the functions f_{ij} are given by

$$
f_{11} = \frac{\sqrt{|\alpha|}}{\alpha}h - \frac{a}{\alpha}, \qquad f_{21} = \mu_2 f_{11} + \eta_2, \qquad f_{31} = \mu_3 f_{11} + \eta_3, \qquad (5.28)
$$

$$
\begin{aligned}
f_{12} = {} & \frac{\sqrt{|\alpha|}}{\alpha}h'z_3 + \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(3h'' + 2\frac{(h')^2}{h}\right)z_1 + \frac{\ell'}{h} - \beta h'\right]z_2 \\
& + \frac{\sqrt{|\alpha|}}{\alpha}\left(h''' + 2\frac{h'h''}{h} - \frac{(h')^3}{h^2}\right)z_1^3 + \frac{\sqrt{|\alpha|}}{\alpha}\left(-\beta h'' - \frac{\beta(h')^2}{h} - \frac{\ell'h'}{h^2} + \frac{\ell''}{h}\right)z_1^2 \\
& + \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(-\epsilon h^2 + a\frac{\sqrt{|\alpha|}}{\alpha}h\right)h' - \beta\frac{\ell'}{h}\right]z_1 - \epsilon\frac{\sqrt{|\alpha|}}{\alpha}\ell h + \frac{a}{|\alpha|}\ell, \qquad (5.29)
\end{aligned}
$$

$$
\begin{aligned}
f_{22} = {} & \mu_2\frac{\sqrt{|\alpha|}}{\alpha}h'z_3 + \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(3\mu_2h'' + 2\mu_2\frac{(h')^2}{h}\right)z_1 - \eta_3h' + \mu_2\frac{\ell'}{h}\right]z_2 \\
& + \frac{\sqrt{|\alpha|}}{\alpha}\left(\mu_2h''' + 2\mu_2\frac{h'h''}{h} - \mu_2\frac{(h')^3}{h^2}\right)z_1^3 \\
& - \frac{\sqrt{|\alpha|}}{\alpha}\left(\eta_3h'' + \eta_3\frac{(h')^2}{h} + \mu_2\frac{\ell'h'}{h^2} - \mu_2\frac{\ell''}{h}\right)z_1^2 \\
& - \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(\epsilon\mu_2h^2 + \frac{\sqrt{|\alpha|}}{\alpha}\kappa h\right)h' + \frac{\ell'\eta_3}{h}\right]z_1 - \epsilon\frac{\sqrt{|\alpha|}}{\alpha}\mu_2\ell h - \frac{\ell\kappa}{|\alpha|}, \qquad (5.30)
\end{aligned}
$$

$$
\begin{aligned}
f_{32} = {} & \mu_3\frac{\sqrt{|\alpha|}}{\alpha}h'z_3 + \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(3\mu_3h'' + 2\mu_3\frac{(h')^2}{h}\right)z_1 - \eta_2h' + \mu_3\frac{\ell'}{h}\right]z_2 \\
& + \frac{\sqrt{|\alpha|}}{\alpha}\left(\mu_3h''' + 2\mu_3\frac{h'h''}{h} - \mu_3\frac{(h')^3}{h^2}\right)z_1^3 \\
& - \frac{\sqrt{|\alpha|}}{\alpha}\left(\eta_2h'' + \eta_2\frac{(h')^2}{h} + \mu_3\frac{\ell'h'}{h^2} - \mu_3\frac{\ell''}{h}\right)z_1^2 \\
& - \frac{\sqrt{|\alpha|}}{\alpha}\left[\left(\epsilon\mu_3h^2 - \frac{\sqrt{|\alpha|}}{\alpha}\gamma h\right)h' + \frac{\ell'\eta_2}{h}\right]z_1 - \epsilon\frac{\sqrt{|\alpha|}}{\alpha}\mu_3\ell h + \frac{\ell\gamma}{|\alpha|}, \qquad (5.31)
\end{aligned}
$$

where a and γ are the constants defined by (3.1),

$$\kappa := \mu_2\mu_3\eta_3 + \left(1-\mu_3^2\right)\eta_2 \tag{5.32}$$

and $\mu_2, \mu_3, \eta_2, \eta_3$ are such that $\epsilon = sgn(\alpha)$ and $\delta_1/\gamma = m$.

Proof. The results are obtained by taking into account (5.1), (5.18), (5.19) and substituting into (4.2), (4.15) and (4.13)–(4.14). Considering α such that $sgn(\alpha) = \epsilon$, we define the function $h = \frac{\alpha f_{11}(z)+a}{\sqrt{|\alpha|}}$ and $\ell = \epsilon r(u)$. Then we obtain (5.27) in terms of h and ℓ and the functions f_{ij} given by (5.28)–(5.31).

Now, we have to verify that the functions f_{ij} satisfy the generic condition (2.8). If $\eta_2 \neq 0$, then the coefficient of z_3 in $f_{11}f_{22} - f_{12}f_{21}$ is nonzero. If $\eta_2 = 0$ then, since $\rho \neq 0$, we have that $\eta_3 \neq 0$, and the coefficient of z_2 reduces to $-\frac{\sqrt{|\alpha|}}{\alpha}\eta_3(1+\mu_2^2)f_{11}h'$ which is nonzero on an open set. Therefore, the generic condition (2.8) is satisfied.

The converse of Theorem 5.6 is a straightforward computation. □

Observe that equation (5.27) coincides with (3.4) and it is called of Type III in our main classification result, Theorem 3.1. It follows from Theorem 5.6 that by choosing a constant $m \neq 0$, $\epsilon = \pm 1$ and arbitrary differentiable functions $h(z)$ and $\ell(z)$, independent of the parameters μ_s, η_s, $s = 2, 3$, such that h' does not vanish on an open set, the differential equation (5.27) is the integrability condition of a linear problem (2.5) determined by the functions f_{ij}, which are given by (5.28)–(5.31) in terms of h, ℓ and 3 parameters, since $\delta_1/\gamma = m$. An example of such an equation is given by Example 3.5, which is obtained by choosing $h = z^2$ and $\ell = 0$ in Theorem 5.6.

6. Case $\rho = 0$

This section is devoted to the complete description of equations that describe pseudospherical surfaces of the form (1.1), with f_{ij} satisfying (1.3), when $\rho = 0$. In the previous section we treated the generic case $\rho \neq 0$, by explicitly solving two equations $B_2 = C_2 = 0$. In spite of some natural differences, the analysis here will be parallel to that of the previous section. We start with a preliminary discussion considering $\rho = 0$. In this case, one cannot solve (4.16) with respect to ϕ_{12}, as in Lemma 5.1. However, the following lemma shows that when $\rho = 0$, (4.16) is equivalent to a system of two equations.

Lemma 6.1. *Under the assumptions of Theorem 4.2, if $\rho = 0$, then (4.16) is equivalent to the system*

$$\begin{aligned}
&\left(1+\mu_3^2\right)\psi_{22,z_1} - \mu_2\mu_3\psi_{32,z_1} - \alpha f_{11}'\left[\eta_2 f_{11} + \eta_2^2\mu_2\right]\left(-\mu_2^2 + \mu_2^2\mu_3^2 - 1 - \mu_3^4 - 2\mu_3^2\right) = 0,\\
&z_1\left[\left(1+\mu_3^2\right)\psi_{22,z} - \mu_2\mu_3\psi_{32,z}\right]\\
&\quad + \left[f_{11} + \frac{\eta_2\mu_2}{1+\mu_2^2+\mu_3^2}\right]\left[2\mu_2\mu_3\psi_{22} - \left(1+\mu_2^2+\mu_3^2\right)\psi_{32}\right] = 0.
\end{aligned} \tag{6.1}$$

Proof. After rewriting $\rho = 0$ as

$$\eta_3 = \frac{2\mu_2\mu_3\eta_2}{1+\mu_2^2+\mu_3^2}, \tag{6.2}$$

(4.16) reduces to

$$B_{21}z_2 + B_{20} = 0,$$

where

$$\begin{aligned}
B_{21} &= \left(1+\mu_3^2\right)\psi_{22,z_1} - \mu_2\mu_3\psi_{32,z_1} - \alpha f'_{11}\left[\eta_2 f_{11} + \eta_2^2\mu_2\right]\left(-\mu_2^2 + \mu_2^2\mu_3^2 - 1 - \mu_3^4 - 2\mu_3^2\right), \\
B_{20} &= z_1\left[\left(1+\mu_3^2\right)\psi_{22,z} - \mu_2\mu_3\psi_{32,z}\right] \\
&\quad + \left[f_{11} + \frac{\eta_2\mu_2}{1+\mu_2^2+\mu_3^2}\right]\left[2\mu_2\mu_3\psi_{22} - \left(1+\mu_2^2+\mu_3^2\right)\psi_{32}\right]
\end{aligned}$$

do not depend on z_2. Hence, $B_2 = 0$ is equivalent to the system of equations $B_{21} = B_{20} = 0$, given by (6.1). □

In view of (4.1), the system (6.1) can be readily integrated leading to a linear algebraic system for the functions ψ_{22} and ψ_{32} whose determinant is δ_2 as given in (3.1). Then, similarly to the previous section, one can distinguish two main cases: $\delta_2 = 0$ and $\delta_2 \neq 0$. These two cases will be treated in Sections 6.1 and 6.2, respectively.

6.1. *Case* $\{\rho = 0, \delta_2 = 0\}$

Theorem 6.2. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$ *satisfying (1.3) and* $\{\rho = 0, \delta_2 = 0\}$ *(*ρ *and* δ_2 *are the constants defined by (3.1)), if, and only if, the differential equation has the form*

$$\begin{aligned}
z_t = z_4 &+ \left(\frac{2h''}{h'}z_1 + h + \frac{\psi_{,z_1}}{h'}\right)z_3 + \left(\frac{h''}{h'} + \frac{\psi_{,z_1z_1}}{h'}\right)z_2^2 \\
&+ \left[\frac{h'''}{h'}z_1^2 + \left(h' + \frac{hh''}{h'} + \frac{2\psi_{,zz_1}}{h'}\right)z_1 + h\frac{\psi_{,z_1}}{h'} + \frac{\psi_{,z}}{h'}\right]z_2 \\
&+ \frac{\psi_{,zz}}{h'}z_1^2 + \left[h\frac{\psi_{,z}}{h'} + \psi\right]z_1,
\end{aligned} \tag{6.3}$$

where $h(z)$ *and* $\psi(z, z_1)$ *are arbitrary differentiable functions such that* $h' \neq 0$ *on a nonempty open set and the functions* f_{ij} *are given by*

$$f_{11} = \frac{\epsilon\sqrt{\mu_3^2 - 1}}{2\mu_3}\left(-h - \frac{\eta_2}{\mu_3}\right), \qquad f_{21} = \frac{\epsilon\left(1+\mu_3^2\right)}{\sqrt{\mu_3^2-1}}f_{11} + \eta_2,$$

$$f_{31}=\mu_3 f_{11}+\frac{\epsilon\eta_2}{\mu_3}\sqrt{\mu_3^2-1}, \tag{6.4}$$

$$f_{12}=f'_{11}z_3+\left(f''_{11}z_1-\frac{\epsilon\sqrt{\mu_3^2-1}}{2\mu_3}\psi_{,z1}-\frac{2\varepsilon\mu_3}{\sqrt{\mu_3^2-1}}f_{11}f'_{11}\right)z_2-\frac{\epsilon\sqrt{\mu_3^2-1}}{2\mu_3}\psi_{,z}z_1+f_{11}\psi, \tag{6.5}$$

$$\begin{aligned}f_{22}={}&\epsilon\frac{(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}f_{11}\psi+\eta_2\psi+\varepsilon\frac{(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}f'_{11}z_3-\frac{(1+\mu_3^2)}{2\mu_3}\psi_{,z}z_1\\&+\left\{\epsilon\frac{(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}f''_{11}z_1-\frac{(1+\mu_3^2)}{2\mu_3}\psi_{,z1}-\left[\frac{2\varepsilon\eta_2\mu_3}{\sqrt{\mu_3^2-1}}+2\mu_3\frac{(1+\mu_3^2)}{\mu_3^2-1}f_{11}\right]f'_{11}\right\}z_2,\end{aligned} \tag{6.6}$$

$$\begin{aligned}f_{32}={}&\mu_3 f_{11}\psi+\frac{\epsilon\eta_2\sqrt{\mu_3^2-1}}{\mu_3}\psi+\mu_3 f'_{11}z_3-\epsilon\frac{\sqrt{\mu_3^2-1}}{2}\psi_{,z}z_1\\&+\left[\mu_3 f''_{11}z_1-\epsilon\frac{\sqrt{\mu_3^2-1}}{2}\psi_{,z1}-\left(2\eta_2+\frac{2\varepsilon\mu_3^2}{\sqrt{\mu_3^2-1}}f_{11}\right)f'_{11}\right]z_2,\end{aligned} \tag{6.7}$$

with $\varepsilon=\pm1$, $|\mu_3|>1$, *and* $\eta_2\neq0$.

Proof. By hypothesis, $\rho=0$ and $\delta_2=0$, hence these conditions can be rewritten as

$$\eta_3=\frac{2\mu_2\mu_3\eta_2}{1+\mu_2^2+\mu_3^2},\qquad \mu_2^2=\frac{(1+\mu_3^2)^2}{\mu_3^2-1}, \tag{6.8}$$

hence in particular one has $|\mu_3|>1$ and $\mu_2\neq0$. Then, by integrating the first equation of (6.1), one has

$$\psi_{22}=\frac{\mu_2\mu_3\psi_{32}}{1+\mu_3^2}+\frac{r}{1+\mu_3^2}, \tag{6.9}$$

with $r(z)$ an arbitrary differentiable function. Substituting into the second equation of (6.1), one gets

$$z_1r'+\frac{2\mu_2\mu_3 r f_{11}}{(1+\mu_3^2)}+\frac{2\mu_2^2\mu_3\eta_2 r}{(1+\mu_3^2)(1+\mu_2^2+\mu_3^2)}=0. \tag{6.10}$$

However, since both r and f_{11} do not depend on z_1, and f_{11} cannot be constant, (6.10) entails that

$$r=r_0=const.,\qquad r_0\mu_2\mu_3=0.$$

But, since neither μ_2 nor μ_3 can be zero, the last condition implies that $r_0 = 0$. Hence, when $\delta_2 = 0$, ψ_{22} given by (6.9) and $r = 0$ solve equation (6.1). So, one has to solve only equation (4.17), which now takes the form:

$$C_{21}\phi_{12} + C_{20} = 0, \tag{6.11}$$

where

$$C_{21} = \frac{2\eta_2}{1+\mu_3^2},$$

$$C_{20} = \left(\frac{4\eta_2\mu_2\mu_3 f_{11} f_{11}'}{(1+\mu_3^2)^2} - \frac{2\eta_2 f_{11}'' z_1}{1+\mu_3^2} + \frac{\psi_{32,z_1}}{1+\mu_3^2}\right) z_2 + \frac{\psi_{32,z}}{1+\mu_3^2} z_1 - \frac{2\mu_2\mu_3 f_{11}\psi_{32}}{(1+\mu_3^2)^2},$$

and μ_2 satisfies (6.8). In fact these expressions are obtained by substituting equations (4.13), (4.14), (4.15), (6.8), (6.9) with $r = 0$, and $f_{s1} = \mu_s f_{11} + \eta_s$ into the third equation of (4.12). Moreover, one uses also the fact that, modulo $\delta_2 = 0$, one has the following two identities:

$$\frac{(1+\mu_2^2+\mu_3^2)}{(1+\mu_2^2-\mu_3^2)(1+\mu_3^2)} = \frac{1}{2}, \qquad \frac{\mu_2^2-1-\mu_3^2}{1+\mu_2^2+\mu_3^2} = \frac{1}{\mu_3^2}.$$

Notice that it follows from (6.8) that $1+\mu_2^2-\mu_3^2 = 4\mu_3^2/(\mu_3^2-1) \neq 0$, since $|\mu_3| > 1$.

We claim that $\eta_2 \neq 0$. In fact, otherwise, if $\eta_2 = 0$, since $\psi_{32} = \psi_{32}(z, z_1)$ and $f_{11} = f_{11}(z)$, then $C_{20} = 0$ is equivalent to $\psi_{32} = 0$. Then, (6.9) and $r = 0$ imply that $\psi_{22} = 0$. Moreover, (6.8) and $\eta_2 = 0$ imply that $\eta_3 = 0$ and hence $a = 0$. Then (4.15) implies that $\phi_{22} = \mu_2\phi_{12}$ and from (4.2) we have that

$$f_{22} = z_3 f_{21,z} + \mu_2\phi_{12}, \qquad f_{12} = \mu_3 f_{11}' + \phi_{12}.$$

Since $f_{21} = \mu_2 f_{11}$, we conclude that condition (2.8) would not be satisfied. Therefore, $\eta_2 \neq 0$.

It follows from (6.8) that

$$\mu_2 = \varepsilon\frac{(1+\mu_3^2)}{\sqrt{\mu_3^2-1}}, \qquad \eta_3 = \frac{\varepsilon\eta_2}{\mu_3}\sqrt{\mu_3^2-1}, \qquad \varepsilon = \pm 1.$$

Then, solving (6.11) with respect to ϕ_{12}, one gets

$$\phi_{12} = \left[z_1 f_{11}'' - \frac{2\varepsilon\mu_3}{\sqrt{\mu_3^2-1}} f_{11} f_{11}' - \frac{1}{2\eta_2}\psi_{32,z_1}\right] z_2 - \frac{1}{2\eta_2}\psi_{32,z} z_1 + \frac{\varepsilon\mu_3 f_{11}\psi_{32}}{\eta_2\sqrt{\mu_3^2-1}}. \tag{6.12}$$

Then, the differential equation and the functions f_{ij} follow by substituting the above results into (4.2), (4.3), (4.15) and (4.13)–(4.14). The equation is given by

$$\begin{aligned} z_t = {} & z_4 + \left(-\frac{\psi_{32,z_1}}{2\eta_2 f'_{11}} - \frac{2\varepsilon\mu_3 f_{11}}{\sqrt{\mu_3^2-1}} + \frac{2f''_{11}}{f'_{11}} z_1 - \frac{\eta_2}{\mu_3}\right) z_3 + \left(-\frac{\psi_{32,z_1z_1}}{2\eta_2 f'_{11}} + \frac{f''_{11}}{f'_{11}}\right) z_2^2 \\ & + \left[\frac{f'''_{11}}{f'_{11}} z_1^2 + \left(-\frac{\psi_{32,zz_1}}{\eta_2 f'_{11}} - \frac{2\varepsilon\mu_3}{\sqrt{\mu_3^2-1}} f'_{11} - \frac{2\varepsilon\mu_3}{\sqrt{\mu_3^2-1}} \frac{f_{11}f''_{11}}{f'_{11}} - \frac{\eta_2}{\mu_3}\frac{f''_{11}}{f'_{11}}\right) z_1 \right. \\ & \left. + \left(\frac{\varepsilon\mu_3 f_{11}}{\eta_2\sqrt{\mu_3^2-1}} + \frac{1}{2\mu_3}\right)\frac{\psi_{32,z_1}}{f'_{11}} - \frac{\psi_{32,z}}{2\eta_2 f'_{11}}\right] z_2 \\ & - \frac{\psi_{32,zz}}{2\eta_2 f'_{11}} z_1^2 + \left[\left(\frac{\varepsilon\mu_3 f_{11}}{\eta_2\sqrt{\mu_3^2-1}} + \frac{1}{2\mu_3}\right)\frac{\psi_{32,z}}{f'_{11}} + \frac{\varepsilon\mu_3\psi_{32}}{\eta_2\sqrt{\mu_3^2-1}}\right] z_1. \end{aligned} \tag{6.13}$$

Introducing the function $h = -\frac{2\epsilon\mu_3 f_{11}}{\sqrt{\mu_3^2-1}} - \frac{\eta_2}{\mu_3}$ and $\psi = \frac{\epsilon\mu_3\psi_{32}}{\eta_2\sqrt{\mu_3^2-1}}$ one reduces (6.13) to (6.3) and the functions f_{ij} are given by (6.4)–(6.7).

Now, we have to verify that the functions f_{ij} satisfy the generic condition (2.8). Since $\eta_2 \neq 0$, it follows that the coefficient of z_3 in $f_{11}f_{22} - f_{12}f_{21}$ is nonzero and hence the generic condition (2.8) is satisfied.

The proof of the converse is a straightforward computation. □

In view of Theorem 6.2 one can see that, when $h(z)$ and $\psi(z, z_1)$ are chosen as functions independent of any parameter then equation (6.3) is independent of the parameters, while the corresponding linear problem depends on μ_3 and η_2. An example of such an equation is given by Example 3.6, which is obtained by choosing $h = -z$, $\psi = 0$ and $\eta_2 = \mu_3$, in Theorem 6.2.

We observe that equation (6.3) coincides with (5.7), which we call equation of Type I. However, the corresponding linear problems generated by the functions f_{ij} given in Theorems 5.3 and 6.2 may be distinct.

6.2. *Case* $\{\rho = 0, \delta_2 \neq 0\}$

We start with the following

Lemma 6.3. *If* $\rho = 0$ *and* $\delta_2 \neq 0$, *then the system* (6.1) *is equivalent to*

$$\begin{aligned} \psi_{22} = {} & -\mu_2\eta_2\mu_3\left(1+\mu_2^2-\mu_3^2\right)\left[\frac{f''_{11}}{1+\mu_2^2+\mu_3^2} + \frac{(f'_{11})^2}{(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2}\right] z_1^2 \\ & + \left[\frac{\mu_2\mu_3 r'}{(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2} - \eta_2\left(1+\mu_2^2-\mu_3^2\right)\left(f_{11} + \frac{\mu_2\eta_2}{1+\mu_2^2+\mu_3^2}\right) f'_{11}\right] z_1 + r, \end{aligned} \tag{6.14}$$

and

$$\psi_{32} = -\eta_2\left(1+\mu_3^2\right)\left(1+\mu_2^2-\mu_3^2\right)\left[\frac{f''_{11}}{1+\mu_2^2+\mu_3^2} + \frac{(f'_{11})^2}{(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2}\right] z_1^2$$

$$+\left[\frac{(1+\mu_3^2)r'}{(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2}\right.$$
$$\left.-\frac{2\mu_2\mu_3\eta_2(1+\mu_2^2-\mu_3^2)}{1+\mu_2^2+\mu_3^2}\left(f_{11}+\frac{\eta_2\mu_2}{1+\mu_2^2+\mu_3^2}\right)f'_{11}\right]z_1$$
$$+\frac{2\mu_2\mu_3}{1+\mu_2^2+\mu_3^2}r, \tag{6.15}$$

where $r=r(z)$ *is a differentiable function.*

Proof. Observe that the integration of (6.1) leads to a linear algebraic system for the functions ψ_{22} and ψ_{32}, whose solutions can be written as in (6.14) and (6.15). □

In the remainder of this section, we will use the solutions (6.14)–(6.15) given by this lemma in order to solve equation (4.17). Substituting (6.14)–(6.15) into (4.17), one gets an equation of the form $C_{21}\phi_{12}+C_{20}=0$, with C_{21} and C_{20} not depending on ϕ_{12}. In particular the coefficient C_{21} reads

$$C_{21}=(1+\mu_2^2+\mu_3^2)\left[(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2\right]^2\eta_2\alpha. \tag{6.16}$$

Since f_{11} is not constant, we can restrict ourselves to the open set where $(1+\mu_2^2+\mu_3^2)f_{11}+\eta_2\mu_2\neq 0$. We will consider the two cases: $\alpha=0$ and $\alpha\neq 0$ in Theorems 6.4 and 6.5, respectively.

Theorem 6.4. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated* 1*-forms* $\omega_i=f_{i1}dx+f_{i2}dt$ *satisfying (1.3) and* $\{\rho=0,\delta_2\neq 0,\alpha=0\}$ *(*ρ,α *and* δ_2 *are the constants defined by (3.1)) if, and only if, the differential equation has the form*

$$z_t=z_4+\left(\frac{h''}{h'}z_1+\frac{\phi_{,z_2}}{h'}+\lambda\right)z_3+\frac{\phi_{,z_1}}{h'}z_2+\frac{\phi_{,z}}{h'}z_1-\frac{r_0h}{h'}+\lambda\frac{\phi}{h'}, \tag{6.17}$$

where $h(z),\phi(z,z_1,z_2)$ *are arbitrary differentiable functions, such that* $h'\neq 0$ *on a nonempty open set,* r_0 *is an arbitrary constant, such that* $r_0^2+\lambda^2\neq 0$*,* $\epsilon=\pm 1$ *and the functions* f_{ij} *are given by*

$$f_{11}=h,\qquad f_{21}=\mu_2h+\epsilon\lambda\sqrt{1+\mu_2^2},\qquad f_{31}=\epsilon\sqrt{1+\mu_2^2}h+\mu_2\lambda,$$
$$f_{12}=h'z_3+\phi,$$
$$f_{22}=\mu_2(h'z_3+\phi)+\epsilon\sqrt{1+\mu_2^2}r_0,$$
$$f_{32}=\epsilon\sqrt{1+\mu_2^2}(h'z_3+\phi)+\mu_2r_0. \tag{6.18}$$

Proof. Since $\alpha=0$ and $\rho=0$, it follows from (3.1) that $\mu_3^2=1+\mu_2^2\neq 0$, $a=0$,

$$\eta_3=\epsilon\frac{\mu_2\eta_2}{\sqrt{1+\mu_2^2}},\qquad \delta_2=-4(1+\mu_2^2)\neq 0,\qquad \epsilon=\pm 1.$$

Substituting these expressions together with (6.14) and (6.15), into equation (4.17) one gets

$$r'z_2+\left(r''-\frac{2(1+\mu_2^2)r'f_{11}'}{2(1+\mu_2^2)f_{11}+\mu_2\eta_2}\right)z_1^2-\frac{\eta_2 r' z_1}{\mu_3}=0.$$

Hence, r is a constant. By taking $f_{11}=h(z)$, $\phi_{12}=\phi(z,z_1,z_2)$, $r=\mu_3 r_0$, $\mu_3=\epsilon\sqrt{1+\mu_2^2}$ and $\lambda=\frac{\epsilon\eta_2}{\sqrt{1+\mu_2^2}}$, one gets (6.17)–(6.18) by using (4.2), (4.15) and (4.13)–(4.14).

Now, we have to verify that the functions f_{ij} satisfy the generic condition (2.8). Observe that $f_{11}f_{22}-f_{12}f_{21}=\epsilon\sqrt{1+\mu_2^2}\,[r_0 f_{11}-\lambda(f_{11}'z_3+\psi)]$. If $\lambda\neq 0$, then the coefficient of z_3 is nonzero and if $\lambda=0$ then we need $r_0\neq 0$, in order to have the generic condition $f_{11}f_{22}-f_{12}f_{21}\neq 0$ satisfied. Hence $\lambda^2+r_0^2\neq 0$ must hold.

The converse of the theorem follows from a straightforward computation. □

Observe that equation (6.17) coincides with (3.5) and it is called of Type IV in our main classification result, Theorem 3.1. It follows from Theorem 6.4, that by choosing $h(z)$ and $\phi(z,z_1,z_2)$ as arbitrary differentiable functions, independent of any parameter, such that h' does not vanish on an open set, the differential equation (6.17) is the integrability condition of a linear problem (2.5), determined by the functions f_{ij}, which depend on h, ϕ and the parameter μ_2.

An example of such an equation is given by Example 3.7, which is obtained by choosing in Theorem 6.4, $h=z$, $r_0=0$, $\phi=z^2/2$ and $\lambda=1$. See also Example 7.4.

The next theorem corresponds to the generic condition $\delta_2\neq 0$ and $\alpha\neq 0$. As a consequence of this fact, the form of the corresponding equations is more complicated.

Theorem 6.5. *A differential equation of the form (1.1) describes pseudospherical surfaces, with associated* 1*-forms* $\omega_i=f_{i1}dx+f_{i2}dt$, *satisfying (1.3) and* $\{\rho=0,\delta_2\neq 0,\alpha\neq 0\}$ *(*ρ,α *and* δ_2 *are the constants defined by (3.1)) if, and only if, the differential equation has the form*

$$\begin{aligned}
z_t={}&z_4+\left[\left(\frac{4h''}{h'}+\frac{2h'}{h}\right)z_1+\frac{\ell'}{hh'}\right]z_3+\left(\frac{3h''}{h'}+\frac{2h'}{h}\right)z_2^2+\left[\frac{6h'''}{h'}+\frac{10h''}{h}-5\left(\frac{h'}{h}\right)^2\right]z_1^2z_2\\
&+\left(-\frac{3\ell'}{h^2}+\frac{3\ell''}{hh'}\right)z_1z_2+\left(-\epsilon h^2+m\right)z_2\\
&+\left[\frac{h^{(4)}}{h'}+\frac{2h'''}{h}+2\frac{(h'')^2}{hh'}-\frac{5h'h''}{h^2}+2\left(\frac{h'}{h}\right)^3\right]z_1^4\\
&+\left[\left(-\frac{h''}{h^2h'}+\frac{2h'}{h^3}\right)\ell'+\frac{\ell'''}{hh'}-\frac{2\ell''}{h^2}\right]z_1^3+\left[\epsilon h^2\left(-\frac{h''}{h'}-2\frac{h'}{h}\right)+\left(\frac{h''}{h'}+\frac{h'}{h}\right)m\right]z_1^2\\
&+\left[-\epsilon\left(\frac{h\ell'}{h'}+\ell\right)+m\frac{\ell'}{hh'}\right]z_1,
\end{aligned}\tag{6.19}$$

where $h(z)$, $\ell(z)$ *are arbitrary differentiable functions, such that* $h'\neq 0$ *on a nonempty open set,* $\epsilon=\pm 1$, $m\neq 0$ *is a real constant and the functions* f_{ij} *are given by*

$$f_{11}=\frac{1}{\sqrt{|\alpha|}}(h-\mu_2\nu),\qquad f_{21}=\mu_2 h+\eta_2,\qquad f_{31}=\mu_3(h+2\mu_2\nu),\tag{6.20}$$

$$
\begin{aligned}
f_{12} ={}& \frac{h'}{\sqrt{|\alpha|}} z_3 + \frac{1}{\sqrt{|\alpha|}} \left[\left(3h'' + 2\frac{(h')^2}{h} \right) z_1 + \frac{\ell'}{h} + \nu\mu_3\left(\mu_2^2 - 1 - \mu_3^2\right) h' \right] z_2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left(h''' + 2\frac{h'h''}{h} - \frac{(h')^3}{h^2} \right) z_1^3 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left[-\frac{h'\ell'}{h^2} + \frac{\ell''}{h} + \left(\mu_2^2 - 1 - \mu_3^2\right)\nu\mu_3 \left(h'' + \frac{(h')^2}{h} \right) \right] z_1^2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left[\mu_3\nu\left(\mu_2^2 - 1 - \mu_3^2\right)\frac{\ell'}{h} + \epsilon\left(-h^2 + \nu\mu_2 h\sqrt{|\alpha|}\right)h' \right] z_1 - \frac{\epsilon\ell h}{\sqrt{|\alpha|}} \\
&+ \epsilon\nu\mu_2\ell, \qquad (6.21)
\end{aligned}
$$

$$
\begin{aligned}
f_{22} ={}& +\frac{\mu_2 h'}{\sqrt{|\alpha|}} z_3 + \frac{1}{\sqrt{|\alpha|}} \left[\left(3\mu_2 h'' + 2\mu_2\frac{(h')^2}{h} \right) z_1 + \frac{\mu_2\ell'}{h} - 2\nu\mu_2\mu_3 h' \right] z_2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left[\mu_2 h''' + 2\frac{\mu_2 h'h''}{h} - \frac{\mu_2 (h')^3}{h^2} \right] z_1^3 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left[-\frac{\mu_2 h'\ell'}{h^2} - 2\mu_2\mu_3\nu h'' - 2\frac{\mu_2\mu_3\nu(h')^2}{h} + \frac{\mu_2\ell''}{h} \right] z_1^2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left\{ -2\frac{\mu_2\mu_3\nu\ell'}{h} - \epsilon\left[\mu_2 a^2 + \left(1 + \mu_3^2\right)\nu h\sqrt{|\alpha|}\right]h' \right\} z_1 - \epsilon\frac{\mu_2\ell h}{\sqrt{|\alpha|}} - \epsilon\nu\left(1 + \mu_3^2\right)\ell, \\
&\qquad (6.22)
\end{aligned}
$$

$$
\begin{aligned}
f_{32} ={}& \frac{\mu_3 h'}{\sqrt{|\alpha|}} z_3 + \frac{1}{\sqrt{|\alpha|}} \left[\left(3\mu_3 h'' + 2\frac{\mu_3 (h')^2}{h} \right) z_1 + \frac{\mu_3\ell'}{h} - \left(1 + \mu_2^2 + \mu_3^2\right)\nu h' \right] z_2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left(\mu_3 h''' + 2\mu_3\frac{h'h''}{h} - \mu_3\frac{(h')^3}{h^2} \right) z_1^3 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left[-\mu_3\left(\frac{h'\ell'}{h^2} - \frac{\ell''}{h} \right) - \left(1 + \mu_2^2 + \mu_3^2\right)\nu\left(h'' + \frac{(h')^2}{h} \right) \right] z_1^2 \\
&+ \frac{1}{\sqrt{|\alpha|}} \left\{ -\nu\frac{(1 + \mu_2^2 + \mu_3^2)\ell'}{h} - \epsilon\left[\mu_3 h^2 + \nu\mu_2\mu_3 h\sqrt{|\alpha|}\right]h' \right\} z_1 - \epsilon\frac{\mu_3\ell h}{\sqrt{|\alpha|}} \\
&- \epsilon\nu\mu_2\mu_3\ell. \qquad (6.23)
\end{aligned}
$$

with

$$
\nu := \frac{\eta_2}{1 + \mu_2^2 + \mu_3^2}, \qquad \delta_2\nu^2 = m, \qquad sgn(\alpha) = \epsilon. \qquad (6.24)
$$

Proof. One has only to solve equation (4.17) as already observed in the beginning of this section. We first show that $\rho = 0$ i.e., $\eta_3 = 2\mu_2\mu_3\eta_2/(1 + \mu_2^2 + \mu_3^2)$, $\delta_2 \neq 0$ and $\alpha \neq 0$ imply that $\eta_2 \neq 0$. In fact, otherwise, if $\eta_2 = 0$ then $\eta_3 = a = \gamma = 0$. Hence it follows from (6.14) and (6.15) that equation (4.17) simply reads

$$
r' z_2 + \left(r'' - \frac{r' f_{11}'}{f_{11}} \right) z_1^2 - f_{11}^2 \alpha r = 0.
$$

Hence $r' = r = 0$. Therefore (6.14) implies that $\psi_{22} = \psi_{32} = 0$. From (4.15) we have $\phi_{22} = \mu_2\phi_{12}$ and from (4.2) we have that $f_{12} = z_3 f'_{11} + \phi_{12}$, $f_{22} = z_3 f'_{21} + \mu_2\phi_{12}$. Since $f_{21} = \mu_2 f_{11}$ we conclude that (2.8) is not satisfied and we get a contradiction. Hence $\eta_2 \neq 0$.

Now, by substituting (6.14)–(6.15) into (4.17), one gets an equation of the form $C_{21}\phi_{12} + C_{20} = 0$, where C_{21}, given by (6.16), and C_{20} do not contain ϕ_{12}. Since $\alpha \neq 0$, one can solve this equation with respect to ϕ_{12}. Introducing the functions

$$\begin{aligned}\ell(z) &:= -\eta_2\, sgn(\alpha) r(z),\\ h(z) &:= \left(f_{11} + \frac{\mu_2\eta_2}{1+\mu_2^2+\mu_3^2}\right)\sqrt{|\alpha|},\end{aligned}$$

one obtains

$$\begin{aligned}\phi_{12} = {} & \frac{1}{\sqrt{|\alpha|}}\left\{\left[3h'' + \frac{2(h')^2}{h}\right]z_1 + \nu\mu_3\left(\mu_2^2-\mu_3^2-1\right)h' + sgn(\alpha)\frac{\ell'}{h}\right\}z_2\\ & + \frac{1}{\sqrt{|\alpha|}}\left[h''' + \frac{2h'h''}{h} - \frac{(h')^3}{h^2}\right]z_1^3\\ & + \frac{1}{\sqrt{|\alpha|}}\left[\nu\mu_3\left(\mu_2^2-\mu_3^2-1\right)\left(h'' + \frac{(h')^2}{h}\right) - sgn(\alpha)\frac{h'\ell'}{h^2} + sgn(\alpha)\frac{\ell''}{h}\right]z_1^2\\ & + \frac{1}{\sqrt{|\alpha|}}\left\{\left[\frac{\alpha}{\sqrt{|\alpha|}}\nu\mu_2 h - sgn(\alpha)h^2\right]h' + sgn(\alpha)\frac{\nu\mu_3\ell'(\mu_2^2-\mu_3^2-1)}{h}\right\}z_1\\ & - sgn(\alpha)\ell\left(\frac{h}{\sqrt{|\alpha|}} - \mu_2\nu\right), \end{aligned} \tag{6.25}$$

where ν is given by (6.24) and $sgn(\alpha) = \epsilon$. Considering $m = \delta_2\nu^2$, then (6.19)–(6.23) are obtained by using formulas (4.2), (4.15) and (4.13)–(4.14), the expression of η_3 and the relations (6.24). Observe that $\eta_2\delta_2 \neq 0$ imply that $m \neq 0$.

Now, we have to verify that the functions f_{ij} satisfy the generic condition (2.8). Since $\eta_2 \neq 0$, then the coefficient of z_3 in $f_{11}f_{22} - f_{12}f_{21}$ reduces to $-\eta_2 h'/\sqrt{|\alpha|}$ which is nonzero. The converse of the theorem is a straightforward computation. □

It follows from Theorem 6.5 that by choosing $h(z)$ and $\ell(z)$, arbitrary differentiable functions independent of the parameters μ_s, η_s, $s = 2, 3$, such that h' does not vanish on a nonempty open set, the differential equation (5.27) is the integrability condition of a linear problem (2.5) determined by the functions f_{ij}, which are given explicitly in terms of h, ℓ and 2 parameters, since $\rho = 0$ and $\delta_2\nu^2 = m$.

Notice that (5.27) coincides with (6.19), which we call equation of Type III. However, the corresponding linear problems generated by the functions f_{ij} given by Theorems 5.6 and 6.5 may be distinct.

7. Proof of the main theorem and further examples

The proof of Theorem 3.1 follows from the results of Sections 5 and 6, which give the complete and explicit classification of all differential equations of the form $z_t = z_4 + G(z, z_1, z_2, z_3)$, with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, where the functions f_{ij} satisfy $f_{s1} = \mu_s f_{11} + \eta_s$, for $s = 2, 3$.

In what follows, we exhibit some further examples obtained from the classification results given in the previous sections.

Example 7.1. Fourth-order member of the Burgers hierarchy

$$z_t = z_4 + 4zz_3 + 10z_1z_2 + 6z^2z_2 + 12zz_1^2 + 4z^3z_1$$

is another example of Type I, described by Theorem 5.3. Indeed it corresponds to the choice:

$$h = z, \qquad \psi = 3zz_1 + z^3.$$

Then, by choosing $\mu_2 = \mu_3 = 1$ and $\eta_3 = 0$, one gets that $f_{11} = z - \eta_2$, $\psi_{22} = \eta_2(3zz_1 + z^3)$ and using formulas (5.8)–(5.11) one explicitly determines the corresponding 1-forms

$$\begin{aligned}
\omega_1 &= (z - \eta_2)dx + \big[z_3 + (4z - \eta_2)z_2 + 3z_1^2 + \big(6z^2 - 3\eta_2 z\big)z_1 + z^4 - \eta_2 z^3\big]dt, \\
\omega_2 &= zdx + \big[z_3 + 4zz_2 + 3z_1^2 + 6z^2z_1 + z^4\big]dt, \\
\omega_3 &= \omega_1.
\end{aligned}$$

This equation is the integrability condition of a lower triangular linear system given by (2.5).

Example 7.2. Example 3.3 is an equation of Type I with $\omega_2 = \omega_3$. It follows from Theorem 5.3 (resp. Theorem 6.2) that any differential equation of Type I is the integrability condition of a linear problem (2.5) that depends on 3 (resp. 2) parameters. However, one can also choose the linear problem so that $\omega_2 = \omega_3$. In fact, considering $\eta_2 = \eta_3 \neq 0$ and $\mu_2 = \mu_3$, we have $\delta_1 = 0$, and $\rho = -\eta_2 \neq 0$. Then it follows from Theorem 5.3, that such equations are of the form

$$\begin{aligned}
z_t = z_4 &+ \left(2\frac{h''}{h'}z_1 + h + \frac{\psi_{,z_1}}{h'}\right)z_3 + \left(\frac{h''}{h'} + \frac{\psi_{,z_1z_1}}{h'}\right)z_2^2 \\
&+ \frac{h'''}{h'}z_1^2z_2 + \left(h' + \frac{hh''}{h'} + 2\frac{\psi_{,zz_1}}{h'}\right)z_1z_2 + \left(\frac{h\psi_{,z_1}}{h'} + \frac{\psi_{,z}}{h'}\right)z_2 \\
&+ \frac{\psi_{,zz}}{h'}z_1^2 + \left(\frac{h\psi_{,z}}{h'} + \psi\right)z_1,
\end{aligned} \tag{7.1}$$

with corresponding 1-forms

$$\begin{aligned}
\omega_1 &= -hdx + f_{12}dt, \\
\omega_2 &= (-\mu_2 h + \eta_2)dx + f_{22}dt, \\
\omega_3 &= (-\mu_2 h + \eta_2)dx + f_{22}dt
\end{aligned}$$

with

$$f_{12} = -h'z_3 - \big(h''z_1 + hh' + \psi_{,z_1}\big)z_2 - \psi_{,z}z_1 - \psi h,$$
$$f_{22} = -\mu_2 h'z_3 - \big[\mu_2 h''z_1 + (\mu_2 h - \eta_2)h' + \mu_2\psi_{,z_1}\big]z_2 - \mu_2(\psi h + \psi_{,z}z_1) + \eta_2\psi.$$

Example 7.3. Equation

$$z_t = z_4 + z_3 - 2zz_2 - z_1^2 - 3zz_1$$

is another example of Type II, described by Theorem 5.5. Indeed it corresponds to the choice

$$h = z, \qquad r = z, \qquad \eta_3 = \eta_2, \qquad \mu_2 = 0, \qquad \mu_3 = 1.$$

Using formulas (5.22)–(5.24) one can also determine the corresponding 1-forms

$$\omega_1 = -\frac{z}{\eta_2}dx + \left[-\frac{z_3}{\eta_2} + \left(1 - \frac{1}{\eta_2}\right)z_2 + \left(\frac{z}{\eta_2} + 1\right)z_1 + \frac{z^2}{\eta_2}\right]dt,$$
$$\omega_2 = \eta_2 dx + \big[z_2 + (1-\eta_2)z_1 - \eta_2 z\big]dt,$$
$$\omega_3 = \left(\eta_2 - \frac{z}{\eta_2}\right)dx + \left[-\frac{z_3}{\eta_2} + \left(1 - \frac{1}{\eta_2}\right)z_2 + \left(\frac{z}{\eta_2} + 1 - \eta_2\right)z_1 + \frac{z^2}{\eta_2} - \eta_2 z\right]dt.$$

Example 7.4. The following example is an equation of Type IV which we call a modified Kuramoto–Sivashinsky equation

$$z_t = z_4 + m_1 z_3 + m_2 z_2 - zz_1 + m_0 z^2, \tag{7.2}$$

$m_i, m_0 \in \mathbb{R}$, $m_0 \neq 0$. This equation is obtained from Theorem 6.4 by choosing

$$h = z, \qquad \phi = (m_1 + 2m_0)z_2 + Bz_1 - z^2/2 + 2m_0 Bz,$$

where $B = 4m_0^2 + 2m_0 m_1 + m_2$ and $r_0 = -4m_0 B$.

The corresponding 1-forms, obtained from the theorem, are

$$\omega_1 = zdx + [z_3 + \phi]dt,$$
$$\omega_2 = \left[\mu_2 z - 2m_0\sqrt{1+\mu_2^2}\right]dx + \left[\mu_2(z_3 + \phi) + r_0\sqrt{1+\mu_2^2}\right]dt,$$
$$\omega_3 = \left(\sqrt{1+\mu_2^2}z - 2m_0\mu_2\right)dx + \left[\sqrt{1+\mu_2^2}(z_3 + \phi) + \mu_2 r_0\right]dt.$$

Observe that the Kuramoto–Sivashinsky equation is given by (7.2) when $m_0 = 0$, but it does not describe pseudospherical surfaces, since the one forms do not satisfy (2.8). However, it can be looked as the limiting case of (7.2), when $m_0 \to 0$.

We conclude this section by observing that Example 7.1 is the fourth order member of the Burgers hierarchy. One can show that any equation of this hierarchy describes pseudo-spherical surfaces. In fact, using the approach suggested by Chern and Peng [11] one obtains the following

Theorem 7.5. *The equations of the Burgers hierarchy are given by*

$$z_t = \mathcal{R}^n(z_1), \quad n = 1, 2, ...,$$

where $\mathcal{R}$ is the operator defined as

$$\mathcal{R} = D_x \circ (D_x + z) \circ D_x^{-1},$$

and $\mathcal{R}^n$ denotes the n-th power of $\mathcal{R}$. For each n, the equation describes pseudo-spherical surfaces and the one forms ω_i^n are given by

$$\begin{aligned}
\omega_1^n &:= z dx + \left[-\frac{1}{\eta}(D_x + z) C_n \right] dt, \\
\omega_2^n &:= \eta dx - C_n dt, \\
\omega_3^n &:= -\eta dx + C_n dt,
\end{aligned} \tag{7.3}$$

where

$$C_n = -\eta D_x^{-1} \circ \mathcal{R}^n(z_1).$$

The first equations of the Burgers hierarchy are given by

$$\begin{aligned}
z_t &= z_2 + 2zz_1; \\
z_t &= z_3 + 3zz_2 + 3z_1^2 + 3z^2 z_1; \\
z_t &= z_4 + 4zz_3 + 10z_1 z_2 + 6z^2 z_2 + 12zz_1^2 + 4z^3 z_1; \\
&\dots
\end{aligned}$$

We observe that the second order equation is equivalent to $z_t = z_2 + zz_1$ modulo a scale transformation $z \mapsto 2z$ and it was proven to describe pseudo-spherical surfaces in [12] (the associated 1-forms are the same as in (7.3) modulo the scale transformation).

To the best of our knowledge, it is not yet known a general method to embed a given differential equation that describes a pseudo-spherical surface into an entire hierarchy, if any. An important reference in this aspect is the paper by Reyes [27]. In special cases, as in Theorem 7.5, one may follow the approach given in [11].

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