Existence of optimal controls for SPDE with locally monotone coefficients

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Abstract

The aim of this paper is to investigate the existence of optimal controls for systems described by stochastic partial differential equations (SPDEs) with locally monotones coefficients controlled by different external forces which are feedback controls. We apply our result to various types of SPDEs such as stochastic 2-D Navier-Stokes equation, stochastic nonlocal equation, stochastic linear equations and stochastic semilinear equations.

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1 Introduction

Let H be a real separable Hilbert space. Let V be a reflexive Banach space. Identify H with its dual H' and denote the dual of V by V'. Let

$$V \subset H \cong H' \subset V'$$

where the inclusions are assumed to be dense and compact. The triad (H, V, V') is known as a *Gelfand triple.* We will denote by $\| \cdot \|_V$, $\| \cdot \|$,

$\|\cdot\|_{V'}$ the norms in V, H, and V' respectively. The inner product in H and the duality scalar product between V and V' will be denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ respectively.

Let $\{W_t\}_{t\geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t. a complete filtrered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ and $(L_2(U;H),\|\cdot\|_2)$ denotes the space of all Hilbert-Schmidt operators from U to H.

Let $T>0$ some fixed time, consider the following initial value problem involving a controlled SPDE differential of the form:

$$du(t) = (A(t,u(t),u(t)) + \Phi(t,u(t)))dt + \Xi(t,u(t))dW(t),\ u(0)=u_0 \quad (1)$$

where $A : [0,T]\times V\times V\times\Omega \to V'$, $\Phi : [0,T]\times H\times\Omega \to H$ and $\Xi : [0,T]\times V\times\Omega \to L_2(U;H)$ are progressively measurable, A satisfies a locally monotone condition (see condition **A2** below) and Φ is a control. In this paper we will study the existence of an optimal control which minimizes the cost function $\mathcal{J}(\Phi)$ with Φ belonging to $\mathcal{U}$, the set of controls associated with the controlled initial value problem (1).

The problem of the existence of an optimal control for SPDEs is a common question in optimal control theory and often resolved by assuming that the set of admissible controls is compact and using the Main Theorem for Minimum Problems (see [15, Th. 38.B]). In our work we use a weaker condition to the set of admissible controls which is weak sequentially compact and similarly with the Theorem 38.A of Zeidler [15] we assume that the cost functional is weak sequentially lower semicontinuous. The problem of the existence of an optimal control for SPDEs has been studied by several authors, for example, by Nagase [12], Buckdahn and Răşcanu [2], Gatarek and Sobczyk [5], G. Guisepina and M. Federica [6], and Al-Hussein [7]. The results of these papers cannot be applied in the study of the equation in (1) because they assume semilinearity or boundedness for the nonlinearities. In

[1], the existence of optimal controls for the stochastic Navier - Stokes equation was studied and following the same ideas [4] studied the problem under consideration for a nonlocal parabolic stochastic equation. The present work follows the same ideas and is a generalization for both papers.

The article is organized in the following way: in Section 2, we present the basic spaces, the norms, properties and notations which we are going to work with in the subsequent sections. In Section 3, we formulate the control problem, which is the goal of this work and we prove the existence of an optimal control. The idea is to prove that a minimizing sequence has a subsequence which converges weakly (see Lemma 3.1). Then, we prove that weak convergence of the feedback controls implies strong convergence of a subsequence of the corresponding solutions (see Theorems 3.1 and 3.2). Finally, in Section 4 we provide examples where our result is applied to stochastic 2-D Navier-Stokes equation , stochastic nonlocal equation and stochastic semilinear equation.

2 Preliminaries

To simplify notation we use the letter $\mathbb{T}$ for the interval $[0,T]$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, $(\mathcal{F}_t)_{t\in\mathbb{T}}$ a right-continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}-$null sets and let $\mathbb{E}(X)$ denote the mathematical expectation of the random variable X. We abbreviate "*almost surely* $\omega\in\Omega$." to a.s.

Let B be a Banach space with norm $\|\cdot\|_B$ and let $\mathcal{B}(B)$ denote the Borel $\sigma-$algebra of B. The space $L^2(\Omega\times\mathbb{T};B)$ is the set of all $\mathcal{F}\otimes\mathcal{B}(\mathbb{T})-$measurable processes $u:\Omega\times\mathbb{T}\to B$ which are $\mathcal{F}_t-$ adapted and $\mathbb{E}(\int_{\mathbb{T}}\|u\|_B^2dt)<\infty$. The constant c_{HV} is such that $\|v\|^2\leq c_{HV}\|v\|_V^2$ for all $v\in V$.

In order to get solutions to (1), we state the following conditions on the coefficients: Suppose there exist constants $\alpha > 1$, $\beta \geq 0$, $\theta > 0$, $K > 0$ and a positive adapted process $f \in L^1([0,T] \times \Omega; \mathbb{R})$ such that the following conditions hold for all v, v_1, $v_2 \in V$ and a.e. $(t, \omega) \in \mathbb{T} \times \Omega$.

A1) (Hemicontinuity) The map $s \to \langle A(t, v_1 + sv_2, v_1 + sv_2), v\rangle + \langle \Phi(t, v_1 + sv_2), v\rangle$ is continuous on $\mathbb{R}$.

A2) (Local monotonicity)

$$\begin{aligned} 2\langle A(t, v_1, v_1) - A(t, v_2, v_2), v_1 - v_2\rangle + \quad & 2\langle \Phi(t, v_1) - \Phi(t, v_2), v_1 - v_2\rangle + \\ & + \|\Xi(t, v_1) - \Xi(t, v_2)\|_2^2 \leq \\ & \leq (K + \rho(v_2))\|v_1 - v_2\|^2, \end{aligned}$$

where $\rho : V \to [0, +\infty)$ is a mensurable function and locally bounded in V.

A3) (Coercivity)

$$2\langle A(t, v_1, v_1), v_1\rangle + 2\langle \Phi(t, v_1), v_1\rangle + \|\Xi(t, v_1)\|_2^2 + \leq -\theta\|v_1\|_V^2 + K\|v\|^2 + f(t).$$

A4) (Growth)

$$\|A(t, v_1, v_1)\|_{V'}^2 + \|\Phi(t, v_1)\|_{V'}^2 \leq (f(t) + K\|v_1\|_V^2)(1 + \|v_1\|^\beta).$$

In this work, we understand that the stochastic process u_Φ is a solution to the problem in (1) in the following sense.

Definition 2.1 *Let u_0 be a random variable which does not depend on $W(t)$. The stochastic process $(u_\Phi(t))_{t\in\mathbb{T}} \in L^2(\Omega \times \mathbb{T}; V)$, $\mathcal{F}_t-$ adapted, with a.s. sample paths continuous in H, is a solution to (1) if it satisfies the equation:*

$$\begin{aligned} (u_\Phi(t), v) = \quad & (u_0, v) + \int_0^t \langle A(u_\Phi(s), v\rangle \, ds + \int_0^t (\Phi(s, u_\Phi(s)), v) ds + \\ & + \int_0^t (v, (\Xi(s, u_\Phi(s)) dW(s)) \end{aligned} \tag{2}$$

a.s. for all $v \in V$ and $t \in \mathbb{T}$.

Uniqueness means indistinguishability.

We need the following existence of solutions theorem, the result is a particular case of [10, Th. 1.1] but sufficient for our objective.

Theorem 2.1 *Let $u_0 \in L^4(\Omega, V)$. Suppose that (A1) - (A4) is satisfied and there is a constant C such that*

$$\begin{aligned} \|\Phi(t,v)\|_{V'}^2 \quad &+ \|\Xi(t,v)\|_2^2 \leq C(f(t) + \|v\|^2), \quad t \in \mathbb{T},\ v \in V; \\ \rho(v) \leq \quad &C(1 + \|v\|_V^2)(1 + \|v\|^\beta) \quad v \in V. \end{aligned} \tag{3}$$

The problem (2) has a unique solution u_Φ which has a.s. sample paths continuous in H.

Proof: See [10, Th. 1.1] . □

Remark 2.1 *Our definition of the solution agrees with the one given in [10], see [8, Remark 2.1 p.106] or [9, Remark 3.3 p.32].* □

3 Formulation of the control problem and main result

We consider the controlled SPDE (1) controlled by *continuous feedback controls* and we denote by $\mathcal{U} := \left\{\Phi : \mathbb{T} \times L^2(D) \to L^2(D)\right\}$ the set of the admissible controls satisfying:

$$\|\Phi(0,0)\|^2 \leq \eta \quad a.s. \tag{4}$$

and for all $s, t \in \mathbb{T}$, $x, y \in H$

$$\|\Phi(t,x) - \Phi(s,y)\|^2 \leq \lambda|t-s|^2 + \alpha\|x-y\|^2 \quad a.s. \tag{5}$$

where η, λ, α are positive constants.

Furthermore, we will assume that the coefficients of (1) satisfy the following conditions, for all v, v_1, $v_2 \in V$ and a.e.$(t,\omega) \in \mathbb{T} \times \Omega$:

C1) there is a constant $L > 0$ such that

$$\|\Xi(t,v_1) - \Xi(t,v_2)\|_2^2 \le L\|v_1 - v_2\|^2 \text{ and } \|\Xi(t,v_1)\|_2 = 0$$

C2) there are nonnegative constants K_1 and J_1 such that

$$\langle A(t,v,v_1), v_1\rangle \le -K_1\|v_1\|_V^2 + J_1\|v_1\|^2$$

C3) there is a positive constant θ_1 such that

$$\langle A(t,v,v_1) - A(t,v,v_2), v_1 - v_2\rangle \le -\theta_1 \|v_1 - v_2\|_V^2$$

C4) there are nonnegative constants c_1, c_2, c_3, c_4, c_5, c_6 and c_7 such that

$$\begin{aligned}
\langle A(t,v_1,v_2) - A(t,v_3,v_3), v_2 - v_3\rangle &\le c_1\|v_2 - v_1\|_V\|v_1 - v_3\|_V^{1/2} \\
\|v_1 - v_3\|^{1/2} &\ \|v_2 - v_3\|_V^{1/2}\|v_2 - v_3\|^{1/2} + \\
+c_2\|v_1\|\|v_1\|_V &\ \|v_1 - v_2\|_V\|v_1 - v_2\| + \\
(c_7 + c_3\rho(v_1))\|v_2 - v_3\|^2 &\ +c_4\|v_2 - v_1\|_V^2 + c_5\rho(v_1)\|v_1 - v_2\|_V^2 + \\
&\ -c_6\|v_2 - v_3\|_V^2.
\end{aligned}$$

C5) there are nonnegative constants θ_2, p_3, p_4 and p_5 such that

$$\|A(t,v,v_1)\|_{V'}^2 \le \theta_2 \|v_1\|_V^2 + p_3\|v\|^2\|v\|_V^2 + p_4\|v_1\|^2\|v_1\|_V^2 + p_5$$

Remark 3.1 *Under the conditions (4), (5) and (***C1***) the solution u_Φ obtained in the Theorem 2.1 satisfies:*

$$\mathbb{E}(\sup_{t\in\mathbf{T}} \|u_\Phi(t)\|^2) + \mathbb{E}(\int_0^T \|u_\Phi(s)\|_V^2 ds) \le c\,\mathbb{E}(\|u_0\|^2) \tag{6}$$

and

$$\mathbb{E}(\sup_{t\in\mathbf{T}} \|u_\Phi(t)\|^4) + \mathbb{E}\left(\int_0^T \|u_\Phi(s)\|^2 ds\right)^2 \le c\,\mathbb{E}(\|u_0\|^4) \tag{7}$$

where $c = c(L,\eta,\lambda,\alpha,\theta,T)$ is a positive constant.

Let us now define the *cost functional*

$$\mathcal{J}(\Phi) := \mathbb{E}(\int_0^T \Big(\mathcal{L}(s, u_\Phi(s)) + \mathcal{K}(\Phi(s, u_\Phi(s)))\Big) ds) + \mathbb{E}(\mathcal{H}(u_\Phi(T))),\ \ \Phi \in \mathcal{U} \tag{8}$$

whenever the integral in (8) exists and is finite, with $\mathcal{L} : \mathbb{T} \times H_0^1(D) \to \mathbb{R}_+$, $\mathcal{K} : L^2(D) \to \mathbb{R}_+$, and $\mathcal{H} : L^2(D) \to \mathbb{R}_+$. It is required that the mappings $\mathcal{K}$, $\mathcal{H}$, and $u \in L^2(\mathbb{T}; H_0^1(D)) \longmapsto \int_0^T \mathcal{L}(s, u(s))ds$ are weak sequentially lower semicontinuous.

Our control problem is to minimize (8) over $\mathcal{U}$, we denote by $(\mathcal{P})$ the problem of minimizing $\mathcal{J}$ among the admissible controls. Any $\Phi^* \in \mathcal{U}$ satisfying $\mathcal{J}(\Phi^*) = \inf\{J(\Phi) : \Phi \in \mathcal{U}\}$ is called an *optimal control.*

The following lemma proves that given a minimizing sequence for the problem $(\mathcal{P})$ we can obtain a subsequence and a mapping $\Phi \in \mathcal{U}$, such that the subsequnce converges weakly to Φ.

Lemma 3.1 *Let Φ_n be a minimizing sequence for problem $(\mathcal{P})$. There exists a subsequence n_k of n and a mapping $\Phi \in \mathcal{U}$ such that for all $t \in \mathbb{T}$, x, $y \in H$, we have*

$$\lim_{k\to\infty} (\Phi_{n_k}(t, x), y) = (\Phi(t, x), y). \tag{9}$$

Proof: See [1, Lemma 4.1]. ☐

For simplicity the subsequence of $\{\Phi_{n_k}\}_{k=1}^{\infty}$ obtained in the previous lemma will be relabeled as the same, for this sequence and Φ as in the last lemma let us consider the equation

$$\begin{aligned} (\hat{u}_{\Phi_n}(t), v) &= (u_0, v) + \int_0^t \langle A(u_\Phi(s), \hat{u}_{\Phi_n}(s)), v\rangle \, ds + \\ &+ \int_0^t (\Phi_n(s, u_\Phi(s)), v) \, ds + \int_0^t (v, \Xi(s, u_\Phi(s)) dW(s)) \end{aligned} \tag{10}$$

a.s., $v \in V$, $t \in \mathbb{T}$ and for $n \in \mathbb{Z}^+$. Since the coefficients in the equation (10) satisfied the condition (**C2**), (**C3**), (**A1**), (**A3**) and (**A4**), there is a unique process $\hat{u}_{\Phi_n} \in L^2(\Omega \times \mathbb{T}; V)$ which is a solution of (10) with a.s. continuous trajectories in H (see [13, Th. 4.2.5, p. 75] or [8, Th. 2.1, p. 106] or [9, Th. 3.6, p. 32]) satisfying:

$$\mathbb{E}(\sup_{t\in\mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^4)+\mathbb{E}\left(\int_0^T \|\hat{u}_{\Phi_n}(s)\|_1^2 ds\right)^2 \leq c\,(\mathbb{E}(\|u_0\|^4)+\mathbb{E}(\int_0^T \|u_\Phi(s)\|^4 ds)) \tag{11}$$

where c is a positive constant independent of n.

To obtain the estimates in (11) we use the Burkholder and Schwarz inequalities.

Theorem 3.1 *The solution to (2) and (10) satisfies:*

$$\lim_{n\to\infty} \mathbb{E}(\int_0^T \|(u_\Phi - \hat{u}_{\Phi_n})(s)\|_V^2 ds) = \lim_{n\to\infty} \mathbb{E}(\|(u_\phi - \hat{u}_{\Phi_n})(T)\|^2) = 0.$$

Proof: Let us consider the equation

$$(z(t), v) = (u_0, v) + \int_0^t \langle A(u_\Phi(s), z(s)), v\rangle\, ds + \int_0^t (v, \Xi(s, u_\Phi(s))dW(s)) \tag{12}$$

a.s., $v \in V$ and $t \in \mathbb{T}$. By a similar argument as in the case of equation (10), there exists a unique solution $z \in L^2(\Omega \times \mathbb{T}; V)$ of (12), which has a.s. continuous trajectories in H. By using the Gronwall lemma, we get the estimate

$$\mathbb{E}(\sup_{t\in\mathbb{T}} \|z(t)\|^2)+2p\mathbb{E}(\int_0^T \|z(s)\|_V^2 ds) \leq k\left(\mathbb{E}(\|u_0\|^2) + \mathbb{E}(\int_0^T \|u_\Phi(s)\|^2 ds)\right).$$

Then, there exists $k_2(\omega) > 0$ and a.s.,

$$\begin{gathered}\sup_{t\in\mathbb{T}} \|z(t)\|^2 \leq k_2(\omega),\\ \int_0^T \|z(s)\|_V^2 ds \leq k_2(\omega)\end{gathered} \tag{13}$$

and

$$\begin{gathered}\sup_{t\in\mathbb{T}} \|u_\Phi(t)\|^2 \leq k_2(\omega),\\ \int_0^T \|u_\Phi(s)\|_V^2 ds \leq k_2(\omega).\end{gathered} \tag{14}$$

Using stochastic integral properties and (7), we obtain that for all $s, t \in \mathbb{T}$, $t > s$,

$$\mathbb{E}(\| \int_s^t \Xi(r, u_\Phi(r)) dW(r) \|_{V'}^4) \leq c(t-s)^2 E(\|u_0\|^4)$$

As a result of the Kolmogorov continuity test, we get a random variable $\widetilde{H}$ such that

$$\| \int_s^t \Xi(r, u_\Phi(r)) dW(r) \|_{V'}^2 \leq \widetilde{H}(\omega)|t-s|^{2\gamma} \tag{15}$$

a.s. with $0 < \gamma < \frac{1}{4}$ and for every $t, s \in \mathbb{T}$.

Let $\bar{\Omega} \subset \Omega$ with $\mathbb{P}(\bar{\Omega}) = 1$ such that for $\omega \in \bar{\Omega}$ the equations in (2) and (12) are satisfied and, for each $n \in \mathbb{Z}^+$, (10) is also satisfied and the inequalities in (13), (14) and (15) are satisfied.

From (10), (12), (14) and the properties of A (**C3**) and Φ_n, it follows that for $\omega \in \bar{\Omega}$,

$$\begin{aligned} \sup\nolimits_{t\in\mathbb{T}} \|(\hat{u}_{\Phi_n} - z)(t)\|^2 + \quad & \theta_1 \int_0^T \|(\hat{u}_{\Phi_n} - z)(s)\|_V^2 ds \leq \frac{c_{HV}^2 2T(\lambda T^2 + \eta)}{\theta_1} + \\ & + \tfrac{c_{HV}^2 2\alpha}{\theta_1} \int_0^T \|u_{\Phi(s)}\|_V^2 ds \leq k(\omega), \end{aligned}$$

where $k(\omega)$ is independent of n. Hence, for all $n \in \mathbb{Z}^+$, we obtain

$$\sup_{t\in\mathbb{T}} \|\hat{u}_{\Phi_n}(t)\|^2 + p \int_0^T \|\hat{u}_{\Phi_n}(s)\|_V^2 ds \leq k(\omega) \tag{16}$$

for $\omega \in \bar{\Omega}$, where $k(\omega)$ is a positive constant independent of n.

For $\omega \in \bar{\Omega}$, we consider the sequence

$$F(\omega) := \{\hat{u}_{\Phi_n}(\omega, \cdot)\}_{n=1}^\infty,$$

which is bounded because of (16).

From (10), we obtain

$$\begin{aligned} \|\hat{u}_{\Phi_n}(t) - \quad & \hat{u}_{\Phi_n}(s)\|_{V'}^2 \leq \| \int_s^t \Xi(r, u_\Phi(r)) dW(r) \|_{V'}^2 + \\ & + (t-s) \int_s^t \left(\|A(u_\Phi(r), \hat{u}_{\Phi_n}(r))\|_{V'}^2 + \|\Phi_n(r, u_\Phi(r))|\|_{V'}^2 \right) dr, \end{aligned}$$

for each $t, s \in \mathbb{T}$, $t > s$. From this, (15), (16) and the properties of A (**C5**), Φ_n, we get

$$\|\hat{u}_{\Phi_n}(t) - \hat{u}_{\Phi_n}(s)\|_{V'}^2 \leq k(\omega)(t-s) + \widetilde{H}(\omega)(t-s)^{2\gamma}$$

for $\gamma \in (0, \frac{1}{4})$ and where $k(\omega) > 0$ is independent of n.

Consequently, $F(\omega)$ is equi-continuous in $C([0,T], V')$. Now, using Dubinsky's Theorem, (see [14, Th. 4.1, p. 132]), it follows that $F(\omega)$ is relatively compact in $L^2(0,T;H)$. Thus, there exists a subsequence n_k of n and $\hat{u} \in L^2(0,T;H)$ such that

$$\lim_{k\to\infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - \hat{u})(s)\|^2 ds = 0. \tag{17}$$

From (10), (2) and the properties of A (**C3**) we obtain

$$\begin{aligned} \|\hat{u}_{\Phi_{n_k}}(T) - u_\Phi(T)\|^2 \quad &+ 2\theta_1 \int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(t)\|_V^2 dt \leq \\ &\leq \int_0^T \Big(\Phi_{n_k}(t, u_\Phi(t)) - \Phi(t, u_\Phi(t)), (\hat{u}_{\Phi_{n_k}} - \hat{u})(t)\Big)\, dt + \\ &+ \int_0^T (\Phi_{n_k}(t, u_\Phi(t)) - \Phi(t, u_\Phi(t)), (\hat{u} - u)(t))\, dt. \end{aligned}$$

We use Lemma 3.1, (17) and the properties of Φ_n and Φ to obtain

$$\lim_{k\to\infty} \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(T)\|^2 = \lim_{k\to\infty} \int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_\Phi)(t)\|_V^2 dt = 0.$$

Since every subsequence of $(\hat{u}_{\Phi_n}(\omega, \cdot)$ has a subsequence which converges to the same limit $u_{\Phi(\omega,\cdot)}$ in the space $L^2(0,T;V)$, it follows that the sequence $(\hat{u}_{\Phi_n}(\omega, \cdot)$ converges to $u_\Phi(\omega, \cdot)$. Similarly, we can conclude that $(\hat{u}_{\Phi_n}(\omega, T)$ converges to $u_\Phi(\omega, T)$ in H.

From Remark (3.1) and (11), the processes $(\hat{u}_{\Phi_n})_{t\in\mathbb{T}}$ and $(u_\Phi)_{t\in\mathbb{T}}$ are uniformly integrable and thus the theorem follows. □

Let $(Q(t))$ be a $H_0^1(D)-$valued process with

$$\int_0^T \|Q(s)\|_1^2 ds < \infty \text{ and } \sup_{t\in\mathbb{T}} \|Q(t)\|^2 < \infty \; a.s.$$

For each M, a nonnegative integer, we define the following stopping times:

$$\bar{\mathcal{T}}_M^Q := \begin{cases} \inf\left\{t \in \mathbb{T} : \int_0^t \|Q(s)\|_V^2 ds \geq M\right\}, \\ T, \text{ if } \int_0^T \|Q(s)\|_V^2 ds < M, \end{cases}$$

$$\widehat{\mathcal{T}}_M^Q := \begin{cases} \inf\left\{t \in \mathbb{T} : \sup_{t\in\mathbb{T}} \|Q(t)\|^2 \geq M\right\} \\ T, \text{ if } \sup_{t\in\mathbb{T}} \|Q(t))\|^2 < M \end{cases}$$

and $\mathcal{T}_M^Q := min\left\{\bar{\mathcal{T}}_M^Q, \widehat{\mathcal{T}}_M^Q\right\}$.

Let Φ_n and Φ the sequence and the map obtained in the Lemma 3.1, the following theorem asserts that there is a subsequence n_k of n such that he correspondent solutions of (2) $u_{\Phi_{n_k}}$ converge strongly to u_Φ.

Theorem 3.2 *Let* $\{\Phi_n\}_{n\in\mathbb{N}}$ *be as in the last theorem. There is a subsequence* n_k *of* n *such that*

$$\lim_{k\to\infty} \mathbb{E}(\int_0^T \|(u_\Phi - u_{\Phi_{n_k}})(s)\|_1^2 ds) = \lim_{k\to\infty} \mathbb{E}(\|(u_\Phi - u_{\Phi_{n_k}})(T)\|^2) = 0.$$

Proof: For the sake of convenience, we use the abbreviations, $u := u_\Phi$ and $\mathcal{T}_M := \mathcal{T}_M^u$ for $M = 1, 2, \ldots$.
Let $e(t) := exp(\int_0^t -2c_7 - \frac{2\alpha}{c_6} - 2L - 2c_3\rho(u(s))ds)$. As a result of the Itô formula, we get

$$\begin{aligned} &e(\mathcal{T}_M)\|\hat{u}_{\Phi_n}(\mathcal{T}_M) - u_{\Phi_n}(\mathcal{T}_M)\|^2 = \\ &= 2\int_0^{\mathcal{T}_M} e(s)\left\langle A(u(s), \hat{u}_{\Phi_n}(s)) - A(u_{\Phi_n}(s), u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\right\rangle ds = \\ &+2\int_0^{\mathcal{T}_M} e(s)(\Phi_n(s, u(s)) - \Phi_n(s, u_{\Phi_n}(s)), (\hat{u}_{\Phi_n} - u_{\Phi_n})(s))ds+ \\ &+2\int_0^{\mathcal{T}_M} e(s)((\hat{u}_{\Phi_n} - u_{\Phi_n})(s), \Xi(s, u(s)) - \Xi(s, u_{\Phi_n}(s))dW(s))+ \\ &+\int_0^{\mathcal{T}_M} e(s)\|\Xi(s, u(s)) - \Xi(s, u_{\Phi_n}(s)\|_2^2 ds+ \\ &+\int_0^{\mathcal{T}_M} e'(s)\|(\hat{u}_{\Phi_n} - u_{\Phi_n})(s)\|^2 ds. \end{aligned}$$

Using the properties of A (**C4**) Φ_n and Ξ, we get

$$\begin{aligned}
&\mathbb{E}(e(\mathcal{T}_M)\|\hat{u}_{\Phi_n}(\mathcal{T}_M)-u_{\Phi_n}(\mathcal{T}_M)\|^2)+\\
&+c_6\mathbb{E}(\int_0^{\mathcal{T}_M} e(s)\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|_V^2 ds)\leq\\
&\leq \mathbb{E}(\int_0^{\mathcal{T}_M} e'(s)\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|^2 ds)+\\
&+2c_3\mathbb{E}(\int_0^{T} e(s)\rho(u(s))\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|^2 ds)+\\
&+2c_4\mathbb{E}(\int_0^{T} e(s)\|(\hat{u}_{\Phi_n}-u)(s)\|_V^2 ds)+\\
&+2c_5\mathbb{E}(\int_0^{T} e(s)\rho(u(s))\|(\hat{u}_{\Phi_n}-u)(s)\|_V^2 ds)+\\
&+2c_1\left(\mathbb{E}(\int_0^{T}\|(\hat{u}_{\Phi_n}-u)(s)\|_V^2 ds)\right)^{1/2}\\
&\times\left(\mathbb{E}(\int_0^{T}\|(u_{\Phi_n}-u)(s)\|_V^2 ds+\int_0^{T}\|(u_{\Phi_n}-u)(s)\|^2 ds+\right.\\
&\left.+\ \mathbb{E}(\int_0^{T}\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|_V^2\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|^2 ds))\right)^{1/2}+\\
&+c_2 M\left(\mathbb{E}(\int_0^{T}\|(\hat{u}_{\Phi_n}-u)(s)\|_V^2 ds\right)^{1/2}\left(\mathbb{E}(sup_{t\in\mathbb{T}}\|(\hat{u}_{\Phi_n}-u)(s)\|^2)\right)^{1/2}+\\
&+\tfrac{\alpha}{c_6}\mathbb{E}(\int_0^{T} e(s)\|\ (u-u_{\Phi_n})(s))\|^2)+\\
&+L\mathbb{E}(\int_0^{\mathcal{T}_M} e(s)\|(u-u_{\Phi_n})(s)\|^2 ds)+\\
&2c_7\mathbb{E}(\int_0^{T} e(s)\|(\hat{u}_{\Phi_n}-u_{\Phi_n})(s)\|^2 ds)+
\end{aligned}\tag{18}$$

From Theorem 3.1, we can get a subsequence $\left\{\hat{u}_{\Phi_{n_k}}\right\}_{k=1}^{\infty}$ that converges to u a.e. $(\omega, t) \in \Omega \times \mathbf{T}$. Thus, from (18), we obtain

$$\begin{aligned}
E((e(\mathcal{T}_M)\|\hat{u}_{\Phi_{n_k}}(\mathcal{T}_M) - u_{\Phi_{n_k}}(\mathcal{T}_M)\|^2) + c_6 E(\int_0^{\mathcal{T}_M} e(s)\|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|_V^2 ds) \leq \\
\leq 2c_5 E(\int_0^T e(s)\rho(u(s))\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds) + \\
+2c_1 \left(E(\int_0^T \|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds)\right)^{1/2} \\
\times \left(E(\int_0^T \|(u_{\Phi_{n_k}} - u)(s)\|_V^2 ds + \int_0^T \|(u_{\Phi_{n_k}} - u)(s)\|^2 ds + \right. \\
\left. +E(\int_0^T \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|_V^2 \|(\hat{u}_{\Phi_{n_k}} - u_{\Phi_{n_k}})(s)\|^2 ds))\right)^{1/2} + \\
+c_2 M \left(E(\int_0^T \|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds\right)^{1/2} \left(E(sup_{t\in\mathbb{T}}\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|^2)\right)^{1/2} + \\
+(2c_4 + 2L + \tfrac{2\alpha}{c_6})E(\int_0^T e(s)\|(\hat{u}_{\Phi_{n_k}} - u)(s)\|_V^2 ds).
\end{aligned}$$

From this, Theorems 3.1 and the triangle inequality, we obtain

$$\lim_{k\to\infty} E(\int_0^{\mathcal{T}_M} \|(u_\Phi - u_{\Phi_{n_k}})(s)\|_1^2 ds) = \lim_{k\to\infty} E(\|(u_\Phi - u_{\Phi_{n_k}})(\mathcal{T}_M)\|^2) = 0,$$

which implies the desired conclusion. □

Finally, we are in a position to formulate our main result.

Theorem 3.3 *Under the assumptions of Theorem 2.1, if moreover the conditions (***C1***)-(***C2***)) are satisfied, then there exists an optimal control for the problem* $(\mathcal{P})$.

Proof: Let $\{\Phi_n\}$ be a minimizing sequence for the problem $(\mathcal{P})$. We apply Lemma (3.1) and Theorem (3.2) to this sequence. Thus, there exists a subsequence $\{\Phi_{n_k}\}$ of $\{\Phi_n\}$ and $\Phi \in \mathcal{U}$ such that, for all $t \in \mathbb{T}$, $x, y \in L^2(D)$ and a.s. $\omega \in \Omega$, the following hold:

$$\lim_{k\to\infty} \left(\Phi_{n_k}(t, u_{\Phi_{n_k}}), y\right) = (\Phi(t, u_\Phi), y)$$

and

$$\lim_{k\to\infty} \int_0^T \|(u_{\Phi_{n_k}} - u_\phi)(s)\|_V^2 ds = \lim_{k\to\infty} \|(u_{\Phi_{n_k}} - u_\phi)(T)\|^2 = 0.$$

From Theorem (3.2) and the weak sequentially lower semicontinuous properties of $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{H}$, we get

$$\mathcal{J}(\Phi) \leq \liminf_{k\to\infty} \mathcal{J}(\Phi_{n_k}).$$

Since $\{\Phi_n\}$ is a minimizing sequence for the problem $(\mathcal{P})$, $\mathcal{J}(\Phi) = \min_{\lambda\in\mathcal{U}} \mathcal{J}(\lambda)$ and thus $\Phi \in \mathcal{U}$ is an optimal feedback control for problem $(\mathcal{P})$.

4 Examples

Example 4.1 *The main result can be applied to initial value problem involving the linear stochastic evolution equations :*

$$du(t) = (\mathcal{A}(t, u(t)) + \Phi(t, u(t)))dt + \Xi(t, u(t))dW(t), \quad u(0) = u_0 \qquad (19)$$

where $\mathcal{A} : \mathbb{T} \times V \times \Omega \to V'$ *is a linear operator. Furthermore, we will suppose that there are constants* α_1, β_1 *and* γ_1 *such that a.e.* $(t,\omega) \in \mathbb{T} \times \Omega$ *and* $v_1, v_2 \in V$*:*

1) $|\langle \mathcal{A}(t, v_1), v_2\rangle \leq \alpha_1 \|v_1\|_V \|v_2\|_V$

2) $|\langle \mathcal{A}(t, v_1), v_1\rangle \leq -\beta_1 \|v_1\|_V^2 + \gamma_1 \|v_1\|^2$. *Then, there is an optimal control* Φ *which minimizes the cost functional* $\mathcal{J}$ *given by 8 (see [1])*

Proof: Under the conditions (1) , (2) (above), (4), (5) and (**C1**) it is not hard to prove that the coefficients of the equation (19) satisfy the conditions (**A1**), (**A2**) with $K = 2\gamma_1 + \frac{2L}{\theta_1} + \alpha$ and $\rho(v_2) = 0$ and (**A3**) with $\theta = \beta_1$, $K = \gamma_1$ $f \equiv 1$, and (**A4**) with $\beta = 2$, thus from the Theorem 2.1 there is a solution u_Φ to the equation (19).

Taking $A(t, u, v) = \mathcal{A}(v)$ we have that the coefficients of the equation (19) satisfy (**C2**) with $K_1 = \beta_1$, $J_1 = 0$, (**C3**) with $\theta_1 = \beta_1$, (**C4**) with $c_1 = c_2 = c_3 = c_4 = 0 = c_5 = 0$ $c_6 = \beta_1$, $c_7 = \gamma_1$, and (**C5**) with $\theta_2 = \alpha_1$,

$p_3 = p_4 = p_5 = 0$. So that the claim follows from Theorem 3.3. $\square$

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Define

$$V = \{v \in H_0^1(\mathcal{O}, \mathbb{R}^2) : \nabla \cdot v = 0 \, a.e \, in \, \mathcal{O}\},$$

with the norm

$$\|v\|_V := \left(\int_{\mathcal{O}} |\nabla v|^2 dx \right)^{1/2}$$

and H is the closure of V in the following norm

$$\|v\|_H := \left(\int_{\mathcal{O}} |v|^2 dx \right)^{1/2}.$$

The linear operator P_H (Helmhotz-Hodge projection) and $\mathcal{A}$ (Stokes operator with viscosity constant ν) are defined by

$$\begin{aligned} P_H &: L^2(\mathcal{O}, \mathbb{R}^2) \to H \, \text{orthogonal projection;} \\ \mathcal{A} &: H^2(\mathcal{O}, \mathbb{R}^2) \cup V \to H, \quad \mathcal{A}u = \nu P_H \Delta u. \end{aligned}$$

and the nonlinear operator

$$B : \mathcal{D}_B \subset H \times V \to H, \qquad B(u, v) = P_H(u \cdot \nabla v) \tag{20}$$

with notation $B(u) = B(u, u)$.

It is well known then that the Navier-Stokes can be reformulated in the following abstract form

$$\frac{\partial u}{\partial t} = \mathcal{A}u + B(u) + f, \quad \text{in} \quad L^2(0, T; V'), \tag{21}$$

with the initial condition

$$u(0) = u_0 \in H, \tag{22}$$

where $f \in L^2(0, T; V')$ denotes some external force. It is standard that for the Gelfand triple

$$V \subset H = H' \subset V',$$

the following maps

$$\mathcal{A} : V \to V', \quad B : V \times V \to V'$$

are well defined and satisfied

$$\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle, \quad \langle B(u,v), v\rangle = 0 \;\; u, v, w \in V.$$

Now, we consider the following initial value problem stochastic 2-D Navier-Stokes equation

$$du(t) = (\mathcal{A}u(t) + B(u(t)) + \Phi(t, u(t))))dt + \Xi(t, X(t))dW(t), \; u(0) = u_0 \quad (23)$$

where W is a Wiener process in H and $\Phi \in \mathcal{U}$ is a control.

Example 4.2 *(Stochastic 2-D Navier-Stokes equation) There is an optimal control Φ which minimizes the cost functional $\mathcal{J}$ given by 8 (see [1]).*

Proof: In this example we will consider $A(t, u, v) = \mathcal{A}v + B(u, v)$ fot $t \in \mathbb{T}$, $u, v \in V$. First, we will verify that if $u_0 \in L^4(\Omega, H)$ then (23) has a unique solution $u = u_\Phi$. In fact, the hemicontinuity (**A1**) is a consequence that B is a bilinear map.

About (**A2**), we have that (see [11])

$$|\langle B(u,v), w\rangle| \le 2\|w\|_V \|u\|^{1/2}\|u\|_V^{1/2}\|v\|^{/12}\|v\|_V^{1/2} \text{ for all } u, v, w \in V \quad (24)$$

then

$$\begin{aligned} \langle B(u), u - v\rangle = \langle B(u,v), u - v\rangle &= \\ = -\langle B(u - v, u - v), v\rangle + \langle B(v), u - v\rangle &\le \\ \le \tfrac{4}{\nu}\|v\|_V^2\|v - u\|^2 + \tfrac{\nu}{4}\|v - u\|_V^2 + \langle B(v), u - v\rangle & \end{aligned}$$

then we obtain the following inequality (Em [11], Lemma 2.3 is obtained a similar inequality)

$$\langle B(u) - B(v), u - v\rangle \le \frac{4}{\nu}\|v\|_V^2\|v - u\|^2 + \frac{\nu}{4}\|v - u\|_V^2 \quad (25)$$

thus

$$\begin{aligned}2\langle \mathcal{A}u-\mathcal{A}v,u-v\rangle+ & \; 2\langle B(u)-B(v),u-v\rangle+2\langle \Phi(t,u)-\Phi(t,v),u-v\rangle\\ +\|\Xi(t,u)-\Xi(t,v)\|_2^2\leq & \; -\nu\|u-v\|_V^2+\tfrac{8}{\nu}\|v\|_V^2\|v-u\|^2+\\ & \; +L\|v-u\|^2+\tfrac{4\alpha}{\nu}\|v-u\|^2\end{aligned}$$

Hence, we have the local monoticity (**A2**) with $\rho(v)=\frac{2}{\nu}\|v\|_V^2$.

As in ([10]) we can use the inequalities

$$\|u\|_{L^4}^2\leq\sqrt{2}\|u\|_{L^2}\|\nabla u\|_{L^2}\quad u\in H_0^1(\mathcal{O}) \tag{26}$$

and (see [11] Lemma 2.1 and proof of the Lemma 2.2)

$$|\langle B(u),w\rangle|\leq\sqrt{2}\|u\|_{L^4(\mathcal{O};\mathbb{R}^2)}\|w\|_V,\ \text{for}\ \|w\|_V\leq 1$$

to get (**A4**) with $\beta=2$. Analogously (**A3**) is verified using the property $\langle B(u,v),v\rangle=0$. Thus, from Theorem (2.1) there is a unique solution for the equation (23).

Now we proceed to demonstrate (**C2**), (**C3**), (**C4**) and (**C5**) in fact, since

$$\begin{aligned}\langle B(v_1,v_2)-B(v_3,v_3),v_2-v_3\rangle & \; =\langle B(v_1-v_3,v_2-v_1),v_2-v_3\rangle+\\ \langle B(v_1-v_2,v_1),v_2-v_3\rangle+ & \; \langle B(v_2-v_3,v_1),v_2-v_3\rangle\leq\\ \leq 2\|v_2-v_1\|_V\|v_1-v_3\|_V^{1/2}\|v_1-v_3\|^{1/2} & \; \|v_2-v_3\|_V^{1/2}\|v_2-v_3\|^{1/2}+\\ +\tfrac{4}{\nu}\|v_1\|\|v_1\|_V & \; \|v_1-v_2\|_V\|v_1-v_2\|+\\ +\tfrac{4}{\nu}\|v_1\|_V^2\|v_2-v_3\|^2 & \; +\tfrac{\nu}{2}\|v_2-v_3\|_V^2,\end{aligned}$$

(see [1] proof of the Theorem 4.3) thus we have

$$\begin{aligned}2\langle \mathcal{A}v_2-\mathcal{A}v_3,v_2-v_3\rangle+ & \; 2\langle B(v_1,v_2)-B(v_3,v_3),v_2-v_3\rangle\leq\\ \leq 4\|v_2-v_1\|_V\|v_1-v_3\|_V^{1/2}\|v_1-v_3\|^{1/2} & \; \|v_2-v_3\|_V^{1/2}\|v_2-v_3\|^{1/2}+\\ +\tfrac{8}{\nu}\|v_1\|\|v_1\|_V & \; \|v_1-v_2\|_V\|v_1-v_2\|+\\ +\tfrac{8}{\nu}\|v_1\|_V^2\|v_2-v_3\|^2 & \; -\nu\|v_2-v_3\|_V^2,\end{aligned}$$

thus (**C4**) is satisfied with $c_1=4$, $c_2=\frac{8}{\nu}$, $c_3=4$, $c_4=c_5=c_7=0$ and $c_6=\nu$. Since $\langle B(v,v_1),v_1\rangle=0$ we have that (**C2**) is satisfied with $K_1=\nu$

and $J_1 = 0$. To demonstrate (**C3**), observe that

$$\langle B(v, v_1) - B(v, v_2), v_1 - v_2\rangle \;\; = \langle B(v, v_1 - v_2), v_1 - v_2\rangle = 0,$$

hence

$$\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2\rangle + \langle B(v, v_1) - B(v, v_2), v_1 - v_2\rangle \leq -\nu \|v_1 - v_2\|_V^2,$$

thus, (**C3**) is satisfied with $\theta_1 = \nu$. In view of (24) we have, for $v \in V$ fixed

$$\|A(t, v, v_1)\|_V^2{}' \leq 2\nu\|v_1\|_V^2 + 4\|v\|_V^2\|v\|^2 + 4\|v_1\|_V^2\|v_1\|^2 \text{ for } v_1 \in V$$

hence (**C5**) is satisfied with $\theta_2 = 2\nu$, $p_3 = p_4 = 4$ and $p_5 = 0$. And the claim follows from Theorem 3.3. □

Now we will consider the following initial-boundary value problem

$$\begin{cases} du(t) = & (a(\int_D u dx)\Delta u + \Phi(t, u))dt + g(t, u)dW(t) \text{ on } t \in]0, T[\,, \\ & u(x,\ 0) = u_0(x) \text{ on } D \text{ and } u(x, t) = 0 \text{ on } \partial D \times]0, T[\end{cases} \tag{27}$$

where D is a bounded open subset of $\mathbb{R}^n$ with smooth boundary ∂D, $n \geq 1$, $a = a(s)$ is a continuous function with Lipschitz constant L such that $0 < p \leq a(s) \leq P$ where p and P are constants, W is a Wiener process in $L^2(D)$ and $\Phi \in \mathcal{U}$ is a control.

In this case the Gelfand triple

$$V \subset H = H' \subset V',$$

where $V = H_0^1(D)$ and $H = L^2(D)$.

Example 4.3 *(Stochastic nonlocal parabolic equation) There is an optimal control Φ which minimizes the cost functional $\mathcal{J}$ given by 8 (see [4])*

Proof: In this example we will consider $A(t, u, v) = (a(\int_D u dx)\Delta v$ fot $t \in \mathbb{T}$, $u, v \in V$. First, we will verify that if $u_0 \in L^4(\Omega, H)$ then (27) has a unique solution $u = u_\Phi$. In fact, the hemicontinuity (**A1**) is a consequence of properties of a.

About (A2), we have

$$\begin{aligned}\langle A(t,u,u)- \quad & A(t,v,v), u-v\rangle \leq \\ & \leq -\langle a(\textstyle\int_D u(x)dx)\nabla u - a(\int_D v(x)dx)\nabla v, \nabla(u-v)\rangle \leq \\ & -\langle a(\textstyle\int_D u(t,x)dx)(\nabla u - \nabla v), \nabla(u-v)\rangle + \\ & -\langle (a(\textstyle\int_D u(t,x)dx) - a(\int_D v(t,x)dx))\nabla v, \nabla(u-v)\rangle\end{aligned}$$

then

$$\begin{aligned}\langle A(t,u,u) - A(t,v,v), u-v\rangle + & \\ +\tfrac{p}{2}\|\nabla(u-v)\|^2 \leq & \quad \tfrac{C(D)L_1}{2p}\|u-v\|^2\|v\|_V^2.\end{aligned}$$

thus

$$\begin{aligned}2(A(t,u) - A(t,v), u-v\rangle + \quad & 2\langle \Phi(t,u) - \Phi(t,v), u-v\rangle \\ +\|\Xi(t,u) - \Xi(t,v)\|_2^2 \leq \tfrac{C(D)L_1}{p}\|v\|_V^2\|v-u\|^2 + & \\ & +L\|v-u\|^2 + \tfrac{2\alpha}{p}\|v-u\|^2.\end{aligned}$$

Hence, we have the local monoticity (**A2**) with $\rho(v) = \frac{C(D)L_1}{p}\|v\|_V^2$ where $C(D) = 1_D$.

We proceed to demonstrate (**A4**), we have

$$|\langle A(t,u), w\rangle|^2 \leq P\|u\|_V^2 \text{ for } \|w\|_V \leq 1$$

so we have (**A4**) with $\beta = 2$. Similarly, (**A3**) is verified. Thus, from Theorem (2.1) there is a unique solution for the equation (27).

The properties of a provide (**C2**) with $K_1 = p$ and $J_1 = 0$. Using the properties of a we obtain (**C3**) with $\theta_1 = p$. Now, we proceed to demonstrate

(**C4**.) In fact, since

$$
\begin{aligned}
\langle A(v_1, v_2 &) - A(v_3, v_3), (v_2 - v_3)\rangle = \\
&= \langle A(v_1, v_2) - A(v_3, v_2), (v_2 - v_3)\rangle + \\
&+ \langle A(v_3, v_2 - v_3), (v_2 - v_3)\rangle = \\
&- \Big((a(\int_D v_1(x)dx - a(\int_D v_3(x)dx))\nabla v_2 - \nabla v_1, \nabla(v_2 - v_3) \Big) + \\
&- \Big((a(\int_D v_1(x)dx - a(\int_D v_3(x)dx))\nabla v_1, \nabla(v_2 - v_3) \Big) + \\
&+ \langle A(v_3, v_2 - v_3), (v_2 - v_3)\rangle \leq \\
&\leq -\tfrac{3p}{4}\|v_2 - v_3\|_V^2 + \tfrac{2P}{p}\|v_2 - v_1\|_V^2 + \\
&+ \tfrac{4L_1^2 C(D)}{p}\|v_1\|_V^2 \|v_2 - v_3\|^2 + \tfrac{4L_1^2 C(D)}{p}\|v_1\|_V^2 \|v_1 - v_2\|_V^2
\end{aligned}
$$

thus (**C4**) is satisfied with $c_1 = 0$, $c_2 = 0$, $c_3 = 4L_1$, $c_4 = \frac{2P}{p}$, $c_5 = 4L_1$, $c_6 = \frac{3p}{4}$ and $c_7 = 0$. Using the properties of a we obtain (**C5**) with $\theta_2 = P$ and $p_3 = p_4 = p_5 = 0$, and the claim follows from Theorem 3.3. $\square$

Let $D \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary.

Lemma 4.1 *Consider the Gelfand triple*

$$V := H_0^1(D) \subset H := L^2(D) \subset V' := H^{-1}(D)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i(u) D_i u,$$

where f_i, for $i = 1, \dots, d$ are bounded Lipschitz functions on $\mathbb{R}$.

(1) If $d < 3$, there exists a constant K_2 such that

$$2\langle A(u) - A(v), u - v\rangle \leq -\|u - v\|_V^2 + (K_2 + K_2\|v\|_V^2)\|u - v\|_H^2,\ u,\ v \in V.$$

(2) If $d = 3$, there exists a constant K_3 such that

$$2\langle A(u) - A(v), u - v\rangle \leq -\|u - v\|_V^2 + (K_3 + K_3\|v\|_V^4)\|u - v\|_H^2,\ u,\ v \in V.$$

(3) If f_i are independent of u for $i = 1, \dots, d$, i.e.

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i D_i u,$$

then for $d \geq 1$ we have

$$2\langle A(u) - A(v), u - v\rangle \leq -\|u - v\|_V^2 + K_4\|u - v\|_H^2,\ u,\, v \in V.$$

where K_4 is a constant.

Proof: See [10], Lemma 3.1 □

Example 4.4 *(Stochastic semi-linear equations). Let $d \leq 3$ and consider the initial value problem involving the controlled semi-linear stochastic equation*

$$du(t) = (\Delta u(t) + \sum_{i=1}^{d} f_i(u(t))D_i u(t) + \Phi(u(t)))dt + \Xi(u(t))dW(t), \quad u(0) = u_0 \tag{28}$$

where $W(t)$ is a Wiener process on $L^2(D)$, Φ is the control and f_i, are bounded Lipschitz functions on $\mathbb{R}$ for $i = 1, \ldots, d$. Suppose that $|f_i(x)| \leq J < 1$. There is an optimal control Φ for the problem ($\mathcal{P}$).

Proof: We can suppose that all f_i with $i = 1, \ldots, d$ have a Lipschitz constant L_1.

We define the map

$$A(u, v) = \Delta v + \sum_{i=1}^{d} f_i(u(t))D_i v(t), \quad u \in V.$$

The hemicontinuity (**A1**) follows from the continuity of f and Ξ. We give the proof of (**A2**)-(**A4**) only for the case $d = 3$; the case $1 \leq d < 3$ is similar. Therefore, by Lemma 4.1

$$2\langle A(u) - A(v), u - v\rangle + 2\langle \Phi(u) - \Phi(v), u - v\rangle \leq -\frac{1}{2}\|u - v\|_V^2 + (\alpha 2 + K_2\|v\|_V^4)\|u - v\|_H^2$$

for $u, v \in V$. Hence, (**A2**) and (**A3**) is satisfied with $\alpha = 2$ and $\rho(v) = K_2\|v\|_V^4$ We proceed to demonstrate (**A4**), we have that

$$|\langle Au, v\rangle|^2 \leq C\|u\|_V^2\|v\|_V^2$$

so we have (**A4**) with $\beta = 2$. Thus, from Theorem (2.1) there is a unique solution for the equation (28).

Now we proceed to verify (**C2**), (**C3**), (**C4**) and (**C5**). Using the properties of f_i we have that (**C2**) is satisfied with $K_1 = 1 - J$ and $J_1 = 0$. Since

$$\langle A(v, v_1) - A(v, v_2), v_1 - v_2) \rangle \leq -(1 - J)\|v_1 - v_2\|_V^2$$

we have that (**C3**) is satisfied with $\theta_1 = 1 - J$.

Using the properties of f_i we obtain the following inequality

$$\begin{aligned} \langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3) \rangle \leq -\|v_2 - v_3\|_V^2 + \\ \textstyle\sum_{i=1}^d \int_D (f_i(v_1) - f_i(v_3))(D_i v_2 - D_i v_1)(v_2 - v_3)dx + \\ + \textstyle\sum_{i=1}^d \int_D (f_i(v_1) - f_i(v_2))D_i(v_1)(v_2 - v_3)dx + \\ + \textstyle\sum_{i=1}^d \int_D (f_i(v_2) - f_i(v_3))D_i(v_1)(v_2 - v_3)dx + \\ + \textstyle\sum_{i=1}^d \int_D f_i(v_3)(D_i v_2 - D_i v_3)(v_2 - v_3)dx \leq \\ \leq -\|v_2 - v_3\|_V^2 + 2J\|v_2 - v_3\|\|v_2 - v_1\|_V + \\ + L_1\|v_1 - v_2\|_{L^4(D)}\|v_2 - v_3\|_{L^4(D)}\|v_1\|_V + \\ + L_1\|v_2 - v_3\|_{L^4(D)}^2\|v_1\|_V + J\|v_2 - v_3\|\|v_2 - v_3\|_V, \end{aligned} \tag{29}$$

for $v_1, v_2, v_3 \in V$. For $d < 3$, from inequality (26) and from (29) we have

$$\begin{aligned} \langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3) \rangle \leq -\tfrac{1}{4}\|v_2 - v_3\|_V^2 + \\ + \|v_2 - v_3\|^2\|v_1\|^2 + 2J^2\|v_2 - v_3\|^2 + (1 + \tfrac{L_1^4}{2})\|v_2 - v_1\|_V^2 + \\ + \tfrac{1}{2}\|v_1 - v_2\|_V^2\|v_1\|_V^2 + \end{aligned}$$

thus (**C4**) is satisfied with $c_1 = c_2 = 0$, $c_3 = \frac{1}{K_2}$, $c_4 = (1 + \frac{L_1^4}{2})$, $c_5 = \frac{1}{2K_2}$, $c_6 = \frac{1}{4}$ and $c_7 = 2J^2$.

For $d = 3$, from (29), Young's inequality and the following inequality (see [11, p. 34]):

$$\|u\|_{L^4(D)}^4 \leq 4 \leq \|u\|_{L^2(D)}\|\nabla u\|_{L^2(D)}^3 \quad u \in V$$

we get

$$\begin{aligned}\langle A(v_1, v_2) - A(v_3, v_3), v_2 - v_3\rangle \leq -\tfrac{1}{4}\|v_2 - v_3\|_V^2 + \\ +L_1^4 3^3 (\tfrac{1+2^8}{2^6})\|v_2 - v_3\|^2\|v_1\|^4 + 2J^2\|v_2 - v_3\|^2 + (1 + \tfrac{3L_1^4}{4})\|v_2 - v_1\|_V^2 + \\ +\tfrac{1}{4}\|v_1 - v_2\|_V^2\|v_1\|_V^4 +\end{aligned}$$

thus (**C4**) is satisfied with $c_1 = 0$, $c_2 = 0$, $c_3 = \frac{1}{K_3} L_1^4 3^3 (\frac{1+2^8}{2^6})$, $c_4 = 1 + \frac{3L_1^4}{4}$, $c_5 = \frac{1}{4K_3}$, $c_6 = \frac{1}{4}$ and $c_7 = 2J^2$.

Finally, (**C5**) is satisfied with $\theta_2 = 1 + L$, $p_3 = p_4 = 0$ and $p_5 = 1$, and the claim follows from Theorem 3.3. □

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