

DOMINATED SPLITTINGS FOR FLOWS WITH SINGULARITIES

VITOR ARAUJO, ALEXANDER ARBIETO AND LUCIANA SALGADO

ABSTRACT. We obtain sufficient conditions for an invariant splitting over a compact invariant subset of a C^1 flow X_t to be dominated. In particular, we reduce the requirements to obtain sectional hyperbolicity and hyperbolicity.

1. INTRODUCTION

The theory of hyperbolic dynamical systems is one of the main paradigm in dynamics. Developed in the 1960s and 1970s after the work of Smale, Sinai, Ruelle, Bowen [27, 26, 9, 10], among many others, this theory deals with compact invariant sets Λ for diffeomorphisms and flows of closed finite-dimensional manifolds having a hyperbolic splitting of the tangent space. That is, if X is a vector field and X_t is the generated flow then we say that an invariant and compact set Λ is hyperbolic if there exists a continuous splitting of the tangent bundle over Λ , $T_\Lambda M = E^s \oplus E^X \oplus E^u$, where E^X is the direction of the vector field, the subbundles are invariant under the derivative DX_t of the flow X_t

$$DX_t \cdot E_x^* = E_{X_t(x)}^*, \quad x \in \Lambda, \quad t \in \mathbb{R}, \quad * = s, X, u;$$

E^s is uniformly contracted by DX_t and E^u is uniformly expanded: there are $K, \lambda > 0$ so that

$$\|DX_t|_{E_x^s}\| \leq Ke^{-\lambda t}, \quad \|DX_{-t}|_{E_x^u}\| \leq Ke^{-\lambda t}, \quad x \in \Lambda, \quad t \in \mathbb{R}. \quad (1.1)$$

Very strong properties can be deduced from the existence of a such structure; see for instance [13, 25].

Weaker notions of hyperbolicity, like the notions of dominated splitting, partial hyperbolicity, volume hyperbolicity and singular or sectional hyperbolicity (for singular flows) have been proposed to try to enlarge the scope of this theory to classes of systems beyond the uniformly hyperbolic ones; see [5] and [1] for singular or sectional hyperbolicity. However, the existence of dominated splittings is the weaker one.

Many researchers have studied the relations of the existence of dominated splittings with other dynamical phenomena, mostly in the discrete time case, such as robust transitivity, homoclinic tangencies and heteroclinic cycles, and also the possible extension of this notion to endomorphisms; see for instance [20, 33, 21, 19, 5, 6, 15].

However, the notion of dominated splittings deserves attention. Several authors used this notion for the Linear Poincaré flow, see [11, 16, 17], and this is useful in the absence of singularities, as in [12]. We remark that this flow is defined only in the set of regular points. However, in the singular case, it is a non-trivial question to obtain a dominated splitting for the derivative of the flow, and thus allowing singularities. Indeed, it is difficult to obtain it, from a dominated splitting for the Linear Poincaré Flow. See [14], for an attempt to solve this, in the context of robustly transitive sets, using the extended Linear Poincaré Flow.

Definition 1.1. A *dominated splitting* over a compact invariant set Λ of X is a continuous DX_t -invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants K, λ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \quad \text{for all } x \in \Lambda, \text{ and all } t > 0. \quad (1.2)$$

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The purpose of this article is to study the existence of dominated splittings for flows with singularities. In one hand, we study this question in the presence of some sectional hyperbolicity, once that this theory was build to understand flows like the Lorenz attractor, which are the prototype of non-hyperbolic dynamics with singularities, but with robust dynamical properties. On the other hand, we present several examples to clarify the role of the condition on the singularities.

This article is organized as follows. In Section 2, we present definitions and auxiliary results needed in the proof of Theorem A, and some examples showing that the domination condition on the singularities is necessary. In Section 3, we present applications of our main result and more examples implying that these results are not valid for diffeomorphisms. In Section 4 Theorem B is proved assuming Theorem A. In Section 5, we prove the Theorem A and Theorem 2.4. Theorem C and Corollary 2 are proved in Section 6 using the previously proved results. Finally, in Section 7, we present the examples stated in Propositions 1 and 2.

2. STATEMENTS OF THE RESULTS

Let M be a connected compact finite n -dimensional manifold, $n \geq 3$, with or without boundary. We consider a vector field X , such that X is inwardly transverse to the boundary ∂M , if $\partial M \neq \emptyset$. The flow generated by X is denoted by $\{X_t\}$.

An *invariant set* Λ for the flow of X is a subset of M which satisfies $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. The *maximal invariant set of the flow* is $M(X) := \bigcap_{t \geq 0} X_t(M)$, which is clearly a compact invariant set.

A *singularity* for the vector field X is a point $\sigma \in M$ such that $X(\sigma) = \vec{0}$. The set formed by singularities is denoted by $\text{Sing}(X)$. We say that a *singularity is hyperbolic* if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity σ have nonzero real part.

A compact invariant set Λ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle E is *uniformly contracted*, i.e. there exists $C > 0$ and $\lambda > 0$ such that $\|DX_t|_{E_x}\| \leq Ce^{-\lambda t}$ for $t \geq 0$. In this case F is the *central subbundle* of Λ .

We say that a DX_t -invariant subbundle $F \subset T_\Lambda M$ is a *sectionally expanding* subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants C, λ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \text{ for all } t > 0. \quad (2.1)$$

Definition 2.1. [18, Definition 2.7] A *sectional-hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and central subbundle is sectionally expanding.

This is a generalization of hyperbolicity for compact invariant sets with singularities accumulated by regular orbits, since the Lorenz attractor and higher dimensional examples are sectional-hyperbolic [30, 7, 18, 1] and because of the following result whose proof can be found in [19, 5, 1].

Lemma 2.2 (Hyperbolic Lemma). *Every compact invariant subset of a sectional-hyperbolic set without singularities is a hyperbolic set.*

The main result of this paper is the following.

Theorem A. *Let Λ be a compact invariant set of X such that every singularity in this set is hyperbolic. Suppose that there exists a continuous DX_t -invariant splitting of the tangent bundle of Λ , $T_\Lambda M = E \oplus F$, where E is uniformly contracted, F is sectionally expanding and for some constants $C, \lambda > 0$ we have*

$$\|DX_t|_{E_\sigma}\| \cdot \|DX_{-t}|_{F_\sigma}\| < Ce^{-\lambda t} \quad \text{for all } \sigma \in \Lambda \cap \text{Sing}(X) \text{ and } t \geq 0. \quad (2.2)$$

Then $T_\Lambda M = E \oplus F$ is a dominated splitting.

As a corollary, we obtain sufficient conditions to obtain hyperbolicity in the non-singular case using the hyperbolic lemma.

Corollary 1. *Let Λ be a compact invariant set without singularities for the vector field X . Suppose that there exists a continuous DX_t -invariant decomposition of the tangent bundle of Λ , $T_\Lambda M = E \oplus F$, where E contracts and F is sectionally expanding. Then Λ is a hyperbolic set.*

We note that in Definition 2.1 domination is required. As a consequence of Theorem A, the domination assumption is only necessary at the singularities, so that we obtain the following equivalent definition of sectional-hyperbolicity.

Definition 2.3. A compact invariant set $\Lambda \subset M$ is a sectional-hyperbolic set for X if all singularities in Λ are hyperbolic, there exists a continuous DX_t -invariant splitting of the tangent bundle on $T_\Lambda M = E \oplus F$ with constants $C, \lambda > 0$ such that for every $x \in \Lambda$ and every $t > 0$ we have

- (1) $\|DX_t|_{E_x}\| \leq Ce^{-\lambda t}$;
- (2) $|\det DX_t|_{L_x}| > C^{-1}e^{\lambda t}$, for every two-dimensional linear subspace $L_x \subset F_x$;
- (3) $\|DX_t|_{E_\sigma}\| \cdot \|DX_{-t}|_{F_\sigma}\| < Ce^{-\lambda t}$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$.

2.1. Hyperbolicity versus sectional-expansion. We present here some motivation for Theorem A. We observe that, for a hyperbolic set, both splittings $(E^s \oplus E^X) \oplus E^u$ and $E^s \oplus (E^X \oplus E^u)$ are dominated. Moreover, it is easy to see that a hyperbolic set Λ has no singularities accumulated by regular orbits within Λ . Indeed, any singularity σ accumulated by regular orbits on a hyperbolic set Λ would be a discontinuity point for the hyperbolic splitting, due to the absence of the flow direction at σ .

Our next result shows that we cannot have a hyperbolic splitting without the flow direction, unless we restrict ourselves to finitely many singularities, all of them isolated on the nonwandering set.

Theorem 2.4. *Let Λ be a compact invariant set of X . Suppose that there exists a continuous DX_t -invariant decomposition of the tangent bundle $T_\Lambda M = E \oplus F$ over Λ and constants $C, \lambda > 0$, such that for every $x \in \Lambda$ and all $t > 0$*

$$\|DX_t|_{E_x}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t}|_{F_{X_t(x)}}\| \leq Ce^{-\lambda t}.$$

Then Λ consists of finitely many hyperbolic singularities.

We cannot replace the assumptions on Theorem 2.4 either by sectional-expansion or by sectional-contraction, as the following examples show.

Example 1. *Consider a Lorenz-like singularity σ for a C^1 flow $\{X_t\}_{t \in \mathbb{R}}$ on a 3-manifold M , that is, σ is a hyperbolic singularity of saddle-type such that the eigenvalues of $DX(\sigma)$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ and $\lambda_1 + \lambda_2 > 0$.*

Let E_i be the eigenspace associated to the eigenvalue λ_i , $i = 1, 2, 3$, and set $E = E_3$ and $F = E_1 \oplus E_2$. Then the decomposition is trivially continuous, not dominated (F admits vectors more sharply contracted than those of E) but E uniformly contracts lengths of vectors and F uniformly expands area, that is, F is sectionally-expanded.

Example 2. *Consider a hyperbolic saddle singularity, σ , for a C^1 flow $\{X_t\}_{t \in \mathbb{R}}$ on a 4-manifold M such that the eigenvalues of $DX(\sigma)$ are real and satisfy*

$$\lambda_2 < \lambda_3 < 0 < \lambda_4 < \lambda_1, \quad \lambda_1 + \lambda_3 > 0 \quad \text{and} \quad \lambda_2 + \lambda_4 < 0.$$

Let E_i be the eigenspace associated to the eigenvalue λ_i , $i = 1, 2, 3, 4$, and set $F = E_2 \oplus E_4$ and $E = E_1 \oplus E_3$. Then the decomposition is trivially continuous, not dominated as before, E is uniformly area contracting, since $DX_{-t}|_E$ expands area for $t > 0$; and F is uniformly area expanding. In other words, F is sectionally-expanded and E is sectionally-contracted.

In both examples above we have sectional-expansion and sectional-contraction along the subbundles of a continuous splitting but the splitting is not dominated. These examples involve a trivial invariant set: an equilibrium point. But there are examples with compact invariant sets having singularities accumulated by regular orbits and also with a dense regular orbit; see Subsection 7.

This suggests that Theorem 2.4 might be generalized if we assume domination at the singularities of Λ together with sectional-expansion along F and uniform contraction along E over regular orbits. This is precisely the content of Theorem A.

3. APPLICATIONS AND OTHER EXAMPLES

To present the next result characterizing sectional-hyperbolic sets through sectional Lyapunov exponents, we first have to present more definitions.

We say that a probability measure μ is X -invariant if $\mu(X_t(U)) = \mu(U)$, for every measurable subset $U \subset M$ and every $t \in \mathbb{R}$. Given a compact X -invariant set Λ we say that a subset $Y \subset M$ is a *total probability subset* of Λ if $\mu(Y) = 1$ for every X -invariant measure μ supported in Λ . We note that *each singularity σ of Λ belongs to Y* since $\mu = \delta_\sigma$ is a X -invariant probability measure.

Let $A : E \times \mathbb{R} \rightarrow E$ be a Borel measurable map given by a collection of linear bijections

$$A_t(x) : E_x \rightarrow E_{X_t(x)}, \quad x \in M, t \in \mathbb{R},$$

where M is the base space (we assume it is a manifold) of the finite dimensional vector bundle E , satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in M, t, s \in \mathbb{R},$$

with $\{X_t\}_{t \in \mathbb{R}}$ a smooth flow over M . We note that for each fixed $t > 0$ the map $A_t : E \rightarrow E, v_x \in E_x \mapsto A_t(x) \cdot v_x \in E_{X_t(x)}$ is an automorphism of the vector bundle E .

The natural example of a linear multiplicative cocycle over a smooth flow X_t on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle TM of a finite dimensional compact manifold M .

A concrete instance of this structure we will use is the derivative cocycle $A_t(x) = DX_t|_{E_x}$ restricted to a continuous DX_t -invariant subbundle E of the tangent bundle TM .

According to the Multiplicative Ergodic Theorem of Oseledets [3, 4], since in this setting we have that $\sup_{-1 \leq t \leq 1} \log \|A_t(x)\|$ is a bounded function of $x \in \Lambda$, there exists a subset R of Λ with total probability such that for every $x \in R$ there exists a splitting

$$E_x = E_1(x) \oplus \cdots \oplus E_{s(x)}(x) \quad (3.1)$$

which is DX_t -invariant and the following limits, known as the *Lyapunov exponents at x* , exist

$$\lambda_i(x) = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|DX_t(x) \cdot v\|,$$

for every $v \in E_i(x) \setminus \{0\}, i = 1, \dots, s(x)$. We order these numbers as $\lambda_1(x) < \cdots < \lambda_{s(x)}(x)$. One of these subbundles is given by the flow direction (at non-singular points of the flow) and the corresponding Lyapunov exponent is zero for almost every point.

The functions s and λ_i are measurable and invariant under the flow, i.e., $s(X_t(x)) = s(x)$ and $\lambda_i(X_t(x)) = \lambda_i(x)$ for all $x \in R$ and $t \in \mathbb{R}$. The splitting (3.1) also depends measurably on the base point $x \in R$. If F is a measurable subbundle of the tangent bundle then by “the Lyapunov exponents of E ” we mean the Lyapunov exponents of the nonzero vectors in F .

Given a vector space E , we denote by $\wedge^2 E$ the second exterior power of E , defined as follows. For a basis $v_1, \dots, v_n$ of E , then $\wedge^2 E$ is generated by $\{v_i \wedge v_j\}_{i < j}$. Any linear transformation $A : E \rightarrow F$ induces a transformation $\wedge^2 A : \wedge^2 E \rightarrow \wedge^2 F$. Moreover, $v_i \wedge v_j$ can be viewed as the 2-plane generated by v_i and v_j if $i \neq j$; see for instance [3] for more information.

In [2], a notion of sectional Lyapunov exponents was defined and a characterization of sectional-hyperbolicity was obtained based on this notion.

Definition 3.1. [2, Definition 2.2] Given a compact invariant subset Λ of X with a DX_t -invariant splitting $T_\Lambda M = E \oplus F$, the *sectional Lyapunov exponents* of x along F are the limits

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\wedge^2 DX_t(x) \cdot \tilde{v}\|$$

whenever they exists, where $\tilde{v} \in \wedge^2 F_x - \{0\}$.

As explained in [2], if $\{\lambda_i(x)\}_{i=1}^{s(x)}$ are the Lyapunov exponents, then the sectional Lyapunov exponents at a point $x \in R$ are $\{\lambda_i(x) + \lambda_j(x)\}_{1 \leq i < j \leq s(x)}$ and they represent the asymptotic sectional-expansion in the F direction.

As an application of Theorem A, we have the following result, which is an extension of the main result in [2] assuming continuity of the splitting, instead of a dominated splitting.

Theorem B. *Let Λ be a compact invariant set for a X such that every singularity $\sigma \in \Lambda$ is hyperbolic. Suppose that there is a continuous DX_t -invariant splitting $T_\Lambda M = E \oplus F$ such that $T_\sigma M = E_\sigma \oplus F_\sigma$ is dominated, for every singularity $\sigma \in \Lambda$.*

If the Lyapunov exponents in the E direction are negative and the sectional Lyapunov exponents in the F direction are positive on a set of total probability within Λ , then the splitting is dominated and Λ is a sectional hyperbolic set.

We now apply the main result to the setting of weakly dissipative three-dimensional flows.

We recall that an open subset U is a *trapping region* if $\overline{X_t(U)} \subset U, t > 0$ and a compact invariant subset Λ is *attracting* if it is the maximal invariant subset $\Lambda(U) = \bigcap_{t \geq 0} \overline{X_t(U)}$ inside the trapping region U . We say that a compact invariant subset Λ for the flow of the vector field X is *weakly dissipative* if $\text{div}(X)(x) \leq 0$ for all $x \in \Lambda$, that is, the flow near Λ infinitesimally does not expands volume.

We also say that a hyperbolic singularity σ of X is C^1 -linearizable if there exists a diffeomorphism $h : V_\sigma \rightarrow B$ from an open neighborhood of $\sigma \in M$ to an open neighborhood B of the origin in $\mathbb{R}^3$ such that $h(X_t(x)) = e^{tA} \cdot h(x), X_s(x) \in V_\sigma$ for $|s| \leq t$ and $A = DX(\sigma)$. For this it is enough that the spectrum of $DX(\sigma)$ satisfies a finite number of non-resonance conditions; see e.g. [28]. Hence this is a C^r -generic condition for all sufficiently big $r \geq 1$.

Theorem C. *Let X be a C^1 vector field on a three-dimensional manifold M admitting a trapping region U whose singularities (if any) are hyperbolic. Let us assume that the compact invariant subset $\Lambda = \Lambda(U)$ is weakly dissipative and endowed with a one-dimensional continuous field F of asymptotically backward contracting directions, that is, $x \in \Lambda \mapsto F_x$ is continuous and for each $x \in \Lambda$, F_x is a one-dimensional subspace of $T_x M$, and also*

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \log \|DX_{-t} \mid F_x\| < 0 \quad \text{for all } x \in \Lambda. \quad (3.2)$$

If at each $\sigma \in U$ there exists a complementary DX_t -invariant direction E_σ such that $E_\sigma \oplus F_\sigma = T_\sigma M$ is a dominated splitting, then Λ is a hyperbolic set (in particular, Λ has no singularities).

Since we use the attracting and dominated splitting assumption only to prove the non-existence of singularities in Λ , as a consequence of the proof we obtain the following.

Corollary 2. *Let X be a C^1 vector field on a three-dimensional manifold M admitting compact invariant subset Λ , without singularities, which is weakly dissipative and endowed with a one-dimensional continuous field F of asymptotically backward contracting directions.*

Then Λ is a hyperbolic set.

These results are extremely weak versions of the following conjecture of Viana, presented in [31].

Conjecture 1. *If an attracting set $\Lambda(U)$ of smooth map/flow has a non-zero Lyapunov exponent at Lebesgue almost every point of its isolating neighborhood U , that is*

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log \|DX_t\| > 0 \quad \text{for Lebesgue almost every } x \in U, \quad (3.3)$$

then $\Lambda(U)$ has a physical measure: there exists an invariant probability measure μ supported in $\Lambda(U)$ such that for all continuous functions $\varphi : U \rightarrow \mathbb{R}$

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \varphi(X_t(x)) dt = \int \varphi d\mu \quad \text{for Lebesgue almost every } x \in U.$$

Indeed, if U is a trapping region and $\Lambda = \Lambda(U)$ satisfies the assumptions of Theorem C, then Λ is a hyperbolic attracting set, for which it is well-known that (3.3) is true and there exists some physical measure; see e.g. [10, 9, 5].

We now provide several examples which clarify the role of the assumption of domination at the singularities in Theorem A and the relations between dominated splittings for diffeomorphisms and flows.

Proposition 1. *There exist the following examples of vector fields and flows.*

- (1) A vector field with an invariant and compact set containing singularities accumulated by regular orbits inside the set, with a non-dominated and continuous splitting of the tangent space, but satisfying uniform contraction and sectional-expansion.
- (2) A vector field with an invariant and compact set with a continuous and invariant splitting $E \oplus F$ such that E is uniformly contracted, F is area expanding but it is not dominated.
- (3) A suspension flow whose base map has a dominated splitting but the flow does not admit any dominated splitting.

Item (3) above is based on a diffeomorphism described in [8] and similar to another suggested by Pujals in [5, Example B.12].

An important remark is that our results are not valid for diffeomorphisms, as the following result shows.

Proposition 2. *There exists a transitive hyperbolic set for a diffeomorphism with a non-dominated splitting which is sectionally-expanding and sectionally-contracting.*

4. PROOF OF THEOREM B

We follow the lines of [2]. The following proposition, whose proof can be found in [2], is the main auxiliary result in the proof.

We fix a compact X_t -invariant subset Λ . We say that a family of functions $\{f_t : \Lambda \rightarrow \mathbb{R}\}_{t \in \mathbb{R}}$ is sub-additive if for every $x \in M$ and $t, s \in \mathbb{R}$ we have that $f_{t+s}(x) \leq f_s(x) + f_t(X_s(x))$. The Subadditive Ergodic Theorem (see e.g. [32]) shows that the function $\bar{f}(x) = \liminf_{t \rightarrow +\infty} \frac{f_t(x)}{t}$ coincides with $\tilde{f}(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} f_t(x)$ in a set of total probability in Λ .

Proposition 3. *Let $\{t \mapsto f_t : \Lambda \rightarrow \mathbb{R}\}_{t \in \mathbb{R}}$ be a continuous family of continuous function which is subadditive and suppose that $\bar{f}(x) < 0$ in a set of total probability. Then there exist constants $C > 0$ and $\lambda < 0$ such that for every $x \in \Lambda$ and every $t > 0$ we have $\exp(f_t(x)) \leq C \exp(\frac{\lambda t}{2})$.*

Proof. See [2, Proposition 3.4]. $\square$

Remark 1. In [2] only $\Lambda = M(X)$ was considered to conclude that $\{X_t\}_{t \in \mathbb{R}}$ is a sectional-Anosov flow. However all statements are valid for a compact invariant subset of M .

Proof of Theorem B. Define the following families of continuous functions on Λ

$$\phi_t(x) = \log \|DX_t|_{E_x}\| \quad \text{and} \quad \psi_t(x) = \log \|\wedge^2 DX_{-t}|_{F_{X_t(x)}}\|, \quad (x, t) \in \Lambda \times \mathbb{R}.$$

Both families $\phi_t, \psi_t : \Lambda \rightarrow \mathbb{R}$ are easily seen to be subadditive. From Proposition 3 applied to $\phi_t(x)$ and the hypothesis on the Lyapunov exponents, there are constants $C > 0$ and $\gamma < 0$ such that $\exp(\phi_t(x)) = \|DX_t|_{E_x}\| \leq C \exp(\gamma t)$ for all $t > 0$ showing that E is a contractive subbundle.

Analogously, the hypothesis on the sectional Lyapunov exponents and Proposition 3 applied to the function $\psi_t(x)$ provides constants $D > 0$ and $\eta < 0$ for which $\|\wedge^2 DX_{-t}|_{F_{X_t(x)}}\| \leq D e^{\eta t}$, so F is a sectionally expanding subbundle. Now Theorem A ensures that the splitting $E \oplus F$ is a dominated splitting. $\square$

5. PROOF OF THEOREM A AND THEOREM 2.4

We need an auxiliary lemma for which the following definition is necessary. Given a pair of subspaces E_x, F_x of $T_x M$ such that $E_x \cap F_x = \{\vec{0}\}$, the angle $\angle(E_x, F_x)$ between the two subspaces is defined by

$$\sin \angle(E_x, F_x) := \frac{1}{\|\pi(E_x)\|}, \quad x \in \Lambda,$$

where $\pi(E_x) : E_x \oplus F_x \rightarrow E_x$ is the projection onto E_x parallel to F_x defined in the vector space $E_x \oplus F_x$.

We say that an invariant splitting $E \oplus F = T_\Lambda M$ of the tangent bundle over an invariant subset Λ has angle uniformly bounded away from zero if the dimensions of the fibers E_x, F_x are constant for all x in Λ and there exists $\theta_0 > 0$ such that $\sin \angle(E_x, F_x) \geq \theta_0$ for each x in Λ .

The next lemma specifies the subbundle which contains the flow direction.

Lemma 5.1. *Let Λ be a compact invariant set for X . Given an invariant splitting $E \oplus F$ of $T_\Lambda M$ with angle bounded away from zero over Λ , such that E is uniformly contracted, then the flow direction is contained in the F subbundle, for all $x \in \Lambda$.*

Proof. We denote by $\pi(E_x) : T_x M \rightarrow E_x$ the projection on E_x parallel to F_x at $T_x M$, and likewise $\pi(F_x) : T_x M \rightarrow F_x$ is the projection on F_x parallel to E_x . We note that for $x \in \Lambda$

$$X(x) = \pi(E_x) \cdot X(x) + \pi(F_x) \cdot X(x)$$

and for $t \in \mathbb{R}$, by linearity of DX_t and DX_t -invariance of the splitting $E \oplus F$

$$\begin{aligned} DX_t \cdot X(x) &= DX_t \cdot \pi(E_x) \cdot X(x) + DX_t \cdot \pi(F_x) \cdot X(x) \\ &= \pi(E_{X_t(x)}) \cdot DX_t \cdot X(x) + \pi(F_{X_t(x)}) \cdot DX_t \cdot X(x) \end{aligned}$$

Let z be a limit point of the negative orbit of x . That is, we assume without loss of generality since Λ is compact, that there is a strictly increasing sequence $t_n \rightarrow +\infty$ such that $\lim_{n \rightarrow +\infty} x_n := \lim_{n \rightarrow +\infty} X_{-t_n}(x) = z$. Then $z \in \Lambda$ and, if $\pi(E_x) \cdot X(x)$ is not the zero vector, we get

$$\begin{aligned} \lim_{n \rightarrow +\infty} DX_{-t_n} \cdot X(x) &= \lim_{n \rightarrow +\infty} X(x_n) = X(z), \quad \text{but also} \\ \lim_{n \rightarrow +\infty} \|DX_{-t_n} \cdot \pi(E_x) \cdot X(x)\| &\geq \lim_{n \rightarrow +\infty} c^{-1} e^{\lambda t_n} \|\pi(E_x) \cdot X(x)\| = +\infty. \end{aligned} \quad (5.1)$$

This is possible only if the angle between E_{x_n} and F_{x_n} tends to zero when $n \rightarrow +\infty$.

Indeed, using the Riemannian metric on $T_y M$, the angle $\alpha(y) = \alpha(E_y, F_y)$ between E_y and F_y is related to the norm of $\pi(E_y)$ as follows: $\|\pi(E_y)\| = 1/\sin(\alpha(y))$. Therefore

$$\begin{aligned} \|DX_{-t_n} \cdot \pi(E_x) \cdot X(x)\| &= \|\pi(E_{x_n}) \cdot DX_{-t_n} \cdot X(x)\| \\ &\leq \frac{1}{\sin(\alpha(x_n))} \cdot \|DX_{-t_n} \cdot X(x)\| \\ &= \frac{1}{\sin(\alpha(x_n))} \cdot \|X(x_n)\| \end{aligned}$$

for all $n \geq 1$. Hence, if the sequence $\|DX_{-t_n} \cdot \pi(E_x) \cdot X(x)\|$ in the left hand side of (5.1) is unbounded, then $\lim_{n \rightarrow +\infty} \alpha(X_{-t_n}(x)) = 0$.

However, since the splitting $E \oplus F$ has angle bounded away from zero over the compact Λ , we have obtained a contradiction. This contradiction shows that $\pi(E_x) \cdot X(x)$ is always the zero vector and so $X(x) \in F_x$ for all $x \in \Lambda$. $\square$

The proof of Theorem 2.4 follows from Lemma 5.1.

Proof of Theorem 2.4. We assume, arguing by contradiction, that there exists a point $x \in \Lambda \setminus \text{Sing}(X)$, that is, x is a regular point: $X(x) \neq \vec{0}$. Hence Lemma 5.1 ensures that $X(x) \in F_x$. But the same Lemma 5.1 applied to the reversed flow X_{-t} generated by the field $-X$ on the same invariant set Λ shows that $X(x) \in E_x$. Thus $X(x) \in E_x \cap F_x = \{\vec{0}\}$. This contradiction ensures that $\Lambda \subset \text{Sing}(X)$. But our assumptions on the splitting show that each $\sigma \in \Lambda$ is a hyperbolic singularity: the eigenvalues of $DX|_{E_\sigma}$ are negative and the eigenvalues of $DX|_{F_\sigma}$ are positive. It is well-known that hyperbolic singularities σ are isolated in the ambient manifold M ; see e.g. [22]. We conclude that the compact Λ is a finite set of hyperbolic singularities. $\square$

Before proving Theorem A we need some extra notions and definitions.

We recall that, for a invertible continuous linear operator L , the minimal norm (or co-norm) is equal to $m(L) := \|L^{-1}\|^{-1}$. Hence condition (1.2) in Definition 1.1 is equivalent to

$$\|DX_t|_{E_x}\| < K e^{-\lambda t} \cdot m(DX_t|_{F_x}) \text{ for all } x \in \Lambda \text{ and for all } t > 0. \quad (5.2)$$

The Multiplicative Ergodic Theorem of Oseledets [3, 4] provides the following properties of Lyapunov exponents and Lyapunov subspaces, in addition to the existence of Lyapunov exponents on the total probability subset R of Λ , as presented in Section 3. For any pair of disjoint subsets

$I, J \subset \{1, \dots, s(x)\}$, the angle between the bundles $E_I(x) = \bigoplus_{i \in I} E_i(x)$ and $E_J(x) = \bigoplus_{j \in J} E_j(x)$ decreases at most subexponentially along the orbit of x , that is

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sin \angle(E_I(X_t(x)), E_J(X_t(x))) = 0, \quad x \in R;$$

which implies, in particular, that for any pair $i, j \in \{1, \dots, s(x)\}$ with $i \neq j$ and $v_i \in E_i(x) \setminus \{\vec{0}\}, v_j \in E_j(x) \setminus \{\vec{0}\}$

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log |\det(DX_t | \text{span}\{v_i, v_j\})| = \lambda_i(x) + \lambda_j(x).$$

Proof of Theorem A. Let $x \in R \subset \Lambda$ be a regular point of the flow of X . From Lemma 5.1 we know that $X(x) \in F_x$. Since F is a DX_t -invariant subbundle, considering the linear multiplicative cocycle $A_t(z) = DX_t | F_z$ for $z \in \Lambda, t \in \mathbb{R}$, there exists a splitting $F_x = \bigoplus_{j=1}^{s(x)} F_j(x)$ of F_x into a direct sum of Lyapunov subspaces. One of these subspaces is E^X generated by $X(x) \neq \vec{0}$, which we rename $F_1(x) = E_x^X$ in what follows. We also have the corresponding Lyapunov exponents $\lambda_j^F(x), j = 1, \dots, s(x)$.

Fixing $i = 2, \dots, s(x)$ and $v \in F_i(x) \setminus \{\vec{0}\}$, we consider $\text{span}\{X(x), v\}$. From the assumption of sectional expansion of area together with the subexponential control on angles at regular points we obtain

$$0 < \lambda \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |\det(DX_t | \text{span}\{X(x), v\})| = \lambda_1^F(x) + \lambda_i^F(x) = \lambda_i^F(x).$$

Hence $\lambda_i^F(x) \geq \lambda > 0$ for all $i = 2, \dots, s(x), x \in R$ and $X(x) \neq \vec{0}$.

If we consider the cocycle $B_t(z) = DX_t | E_z, x \in \Lambda, t \in \mathbb{R}$, then we find the splitting $E_x = \bigoplus_{i=1}^{r(x)} E_j(x)$ into Lyapunov subspaces and the corresponding Lyapunov exponents $\lambda_i^E(x), i = 1, \dots, r(x)$. It is easy to see that the assumption of uniform contraction along E implies that $\lambda_i^E(x) \leq -\lambda < 0$ for all $x \in R$ and $1 \leq i \leq r(x)$.

Then $\phi_t(x) = \log \frac{\|DX_t|E_x\|}{m(DX_t|F_x)}$ is a subadditive family of continuous functions satisfying

$$\begin{aligned} \bar{\phi}(x) &= \liminf_{t \rightarrow +\infty} \frac{\phi_t(x)}{t} \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \|DX_n|E_x\| - \limsup_{n \rightarrow +\infty} \frac{1}{n} \log m(DX_n|F_x) \\ &= \max\{\lambda_i^E(x), 1 \leq i \leq r(x)\} - \min\{\lambda_i^F(x), 1 \leq i \leq s(x)\} \leq -\lambda - 0 = -\lambda \end{aligned}$$

for all $x \in R$ such that $X(x) \neq \vec{0}$.

For $x = \sigma$ for some singularity $\sigma \in \Lambda$ we have $\bar{\phi}(\sigma) \leq -\lambda$ as a direct consequence of the assumption of domination at the singularities.

We have shown that $\bar{\phi}(x) < 0$ for all $x \in R$. Applying Proposition 3 we see that there exists $C > 0$ such that $\exp \phi_t(x) \leq Ce^{-\lambda t/2}$ for all $x \in \Lambda$ and $t > 0$, which gives us the condition (5.2). This concludes the proof of Theorem A. $\square$

The proof of Corollary 1 follows from Theorem A and the Hyperbolic Lemma 2.2.

6. PROOF OF THEOREM C AND COROLLARY 2

We start by showing that the one-dimensional asymptotically backward contracting subbundle F over Λ is in fact a uniformly expanding subbundle. Indeed, we note that $f_t(x) = \log \|DX_{-t} | F_x\|$ is a subadditive family of continuous functions satisfying $\bar{f}(x) < 0$ for all $x \in \Lambda$. In particular, $\bar{f}(x) < 0$ in a set of total probability. Hence by Proposition 3 we obtain $f_t(x) \leq Ce^{-t\lambda/2}$ for all $t > 0$ and some constants $\lambda, C > 0$. (This trivially implies $\|DX_t | F_{X_t(x)}\| \geq e^{t\lambda/2}/C, x \in \Lambda, t \geq 0$ and so $\|DX_T | F_x\| > 2$ for all $x \in \Lambda$ and $T \geq 2 \log(2C)/\lambda$.)

Now we use the assumption that Λ is an attracting set whose possible singularities σ are hyperbolic, C^1 linearizable and admit a complementary DX_t -invariant direction E_σ such that $E_\sigma \oplus F_\sigma = T_\sigma M$ is a dominated splitting.

Lemma 6.1. *The attracting set Λ does not contain singularities.*

Proof. By the C^1 -linearization assumption and the Stable/Unstable Manifold Theorems, there exists a corresponding X_t -invariant (weak- or strong-)unstable one-dimensional manifold W_σ^u contained in Λ (since Λ is attracting) such that $F_\sigma = T_\sigma W_\sigma^u$ and $X_{-t}(z) \xrightarrow[t \rightarrow +\infty]{} \sigma$ for each $z \in W_\sigma^u$.

Claim 1. *For all $z \in W_\sigma^u$ we have $X(z) \in F_z$.*

We note that this is trivially true for $z = \sigma$. Let us assume, by contradiction, that $X(z) \notin F_z$ for some $z \in W_\sigma^u \setminus \{\sigma\}$ (and thus for all such z by DX_t -invariance). By assumption there exists a diffeomorphism $h : V_\sigma \rightarrow B$ from an open neighborhood of $\sigma \in M$ to an open neighborhood B of the origin in $\mathbb{R}^3$ such that $h(X_t(x)) = e^{tA} \cdot h(x)$, $X_s(x) \in V_\sigma$ for $|s| \leq t$ and $A = DX(\sigma)$. By a linear change of coordinates we can assume without loss of generality that $Dh(\sigma) \cdot F_\sigma = \mathbb{R} \times 0^2$ and $Dh(\sigma) \cdot E_\sigma = 0 \times \mathbb{R}^2$. Let $v : V_\sigma \cap \Lambda \rightarrow \mathbb{R}^3$ denote the continuous unitary vector field such that $\text{span}(v_x) = Dh(x) \cdot F_x$, $x \in V_\sigma \cap \Lambda$.

We are assuming that v_z has a non-zero component along the $E_\sigma \approx 0 \times \mathbb{R}^2$ direction for $z \in W_\sigma^u \setminus \{\sigma\}$. Since $X_{-t}(z) \xrightarrow[t \rightarrow +\infty]{} \sigma$ the domination of the splitting $E_\sigma \oplus F_\sigma$ ensures that $\angle(e^{tA} \cdot v_z, E_\sigma) \xrightarrow[t \rightarrow -\infty]{} 0$. However the invariance of F ensures that $F_{X_{-t}(z)} \xrightarrow[t \rightarrow -\infty]{} F_\sigma$ and since h is a C^1 conjugation we also have $\angle(e^{tA} \cdot v_z, F_\sigma) \xrightarrow[t \rightarrow -\infty]{} 0$. This contradiction proves the claim.

Using the claim together with the previous estimate on expansion along F , we have that

$$\infty > \sup_{x \in \Lambda} \|X(x)\| \geq \|X(X_{kT}(z))\| = \|DX_{kT} \cdot X(z)\| \geq 2^k \|X(z)\|, k \geq 1$$

which is a contradiction and completes the proof of the lemma. $\square$

From now we use only the assumptions of Corollary 2, that is, that Λ is a weakly dissipative compact invariant subset without singularities having a continuous field of asymptotically backward contracting directions.

In this setting, the Linear Poincaré Flow is well-defined over Λ . If we denote by $\mathcal{O}_x : T_x M \rightarrow N_x$ the orthogonal projection from $T_x M$ to $N_x = \{v \in T_x M : \langle v, X(x) \rangle = 0\}$ the orthogonal complement of the field direction at $x \in \Lambda$, then the *Linear Poincaré Flow* is given by $P_x^t = \mathcal{O}_{X_t(x)} \circ DX_t : T_x M \rightarrow N_{X_t(x)}$, $x \in \Lambda, t \in \mathbb{R}$.

We note that since the flow direction is invariant, from the splitting $T_x M = E_x^X \oplus N_x$ we can write $DX_t = \begin{pmatrix} \frac{\|X(X_{-t}(x))\|}{\|X(x)\|} & \star \\ 0 & P^{-t} \end{pmatrix}$ and since $|\det DX_{-t}| \geq 1$ by the weak dissipative assumption, we get $|\det(P^{-t})| \geq m_0 = \min_{z \in \Lambda} \frac{\|X(x)\|}{\|X(X_{-t}(x))\|} > 0$.

Claim 2. *The subbundle $\tilde{F}$ of the normal bundle given by $\{\tilde{F}_x = \mathcal{O}_x(F_x)\}_{x \in \Lambda}$ is P^t -invariant and uniformly backward contracting.*

Indeed, the continuity of the subbundle F ensures that $\angle(F_x, X(x))$ is bounded away from zero, and so there exists $\kappa > 0$ such that for every $v \in \tilde{F}_x$ we have $|\beta| \leq \kappa \|v\|$ such that $\beta X(x) + v = u \in F_x$. This ensures that

$$\begin{aligned} \|P_x^{-t} \cdot v\| &= \|\mathcal{O}_{X_{-t}(x)}(DX_{-t} \cdot (u - \beta X(x)))\| \leq \|\mathcal{O}_{X_{-t}(x)}(DX_{-t} \cdot u - \beta X(X_{-t}(x)))\| \\ &\leq \|DX_{-t} \cdot u\| \leq C e^{-t\lambda/2} \|v + \beta X(x)\| \leq C e^{-t\lambda/2} (\|v\| + \kappa \|v\| \|X(x)\|) \\ &\leq C \left(1 + \kappa \sup_{z \in \Lambda} \|X(z)\|\right) e^{-t\lambda/2} \|v\| = \tilde{C} e^{-t\lambda/2} \|v\|. \end{aligned}$$

This proves Claim 2.

This is enough to guarantee the existence of a complementary P^{-t} -invariant and expanding subbundle through a graph transform technique. To this end, let us consider the one-dimensional subbundle G_x of N_x orthogonal to $\tilde{F}_x$. We can then write $N_x = \tilde{F}_x \oplus G_x$ and so, since $\tilde{F}$ is P^t -invariant, we get $P_x^{-t} = \begin{pmatrix} a_t(x) & b_t(x) \\ 0 & c_t(x) \end{pmatrix}$, where $a_t(x) : \tilde{F}_x \rightarrow \tilde{F}_{X_{-t}(x)}$, $c_t(x) : G_x \rightarrow G_{X_{-t}(x)}$ and $b_t(x) : G_x \rightarrow F_{X_{-t}(x)}$. We know that $|a_t(x)| \leq \tilde{C} e^{-t\lambda/2}$ and since $0 < m_0 \leq |\det(P^{-t})| = |a_t(x)| \cdot |c_t(x)|$ we obtain $|c_t(x)| \geq m_0 e^{t\lambda/2} / \tilde{C}$, $x \in \Lambda, t > 0$. We fix $T > 0$ such that $m_0 e^{T\lambda/2} / \tilde{C} \geq 2$.

Let us now consider the family of continuous one-dimensional complementary subspaces of $\tilde{F}_x$ inside N_x given by the graph of a linear map $L : G_x \rightarrow \tilde{F}_x$, i.e., we consider the vector space $\mathcal{G}$ of all $\mathcal{L} = \{L_x : G_x \rightarrow \tilde{F}_x : x \in \Lambda\}$ with the norm $\|\mathcal{L}^1 - \mathcal{L}^2\| = \sup_{x \in \Lambda} \|L_x^1 - L_x^2\|$. We have that $\mathcal{G}$ with this norm is a Banach space since Λ is compact.

It is easy to see that the map P^{-t} transforms the graph of L_x into the graph of $\hat{L}_{X_{-t}(x)} = (P_x^{-t} \cdot L_x + \pi_{X_{-t}(x)} \cdot P^{-t}) \circ [(I - \pi_{X_{-t}(x)}) \cdot P^{-t}]^{-1}$, where $\pi_z : N_z \rightarrow \tilde{F}_z$ is the orthogonal projection from N_z into $\tilde{F}_z$ at $z \in \Lambda$. In this way we get a map $P^{-T} : \mathcal{G} \rightarrow \mathcal{G}$ which we claim is a contraction with respect to the norm given above. Indeed, for any pair $\mathcal{L}^i = \{L_x^i\}_{x \in \Lambda}$, $i = 1, 2$ we get

$$\begin{aligned} \|\hat{L}_{X_{-T}(x)}^1 - \hat{L}_{X_{-T}(x)}^2\| &= \|P_x^{-T} \cdot (L_x^1 - L_x^2) \circ [(I - \pi_{X_{-T}(x)}) \cdot P^{-T}]^{-1}\| \\ &\leq |a_T(x)| \cdot \|L_x^1 - L_x^2\| / |c_T(x)| \leq \frac{1}{2} \|L_x^1 - L_x^2\| \end{aligned}$$

for each $x \in \Lambda$. We thus have a P^{-T} fixed point $\mathcal{L}^0$ defining a P^{-T} -invariant subbundle $\tilde{E} = \{\tilde{E}_x = \{(u, L_x^0(u)) : u \in G_x\}_{x \in \Lambda}\}$ of the normal bundle. Since P^{-T} commutes with each P^s for all $s \in \mathbb{R}$, we see that $\tilde{E}$ is P^s -invariant for all $s \in \mathbb{R}$.

Claim 3. *The subbundle $E = \tilde{E} \oplus E^X$ is DX_t -invariant.*

Indeed, for any $v = u + \beta X(x)$ with $u \in \tilde{E}_x, \beta \in \mathbb{R}$, we have $DX_t \cdot v = DX_t \cdot u + \beta X(X_t(x))$ and $DX_t \cdot u = P_x^t \cdot u + \gamma X(X_t(x))$ for some $\gamma \in \mathbb{R}$, thus $DX_t \cdot v = P_x^t \cdot u + (\beta + \gamma)X(X_t(x))$ with $P_x^t \cdot u \in \tilde{E}_{X_t(x)}$, hence $DX_t \cdot v \in E_{X_t(x)}$ and the claim is proved.

At this point we have a continuous splitting $T_x M = E_x \oplus F_x, x \in \Lambda$. Using this invariant splitting we can write $DX_{-t} = \begin{pmatrix} A_t(x) & 0 \\ 0 & B_t(x) \end{pmatrix}$ and the continuity of the splitting ensures that $\angle(E_x, F_x)$ is bounded away from zero. This ensures that $1 \leq |\det DX_{-t}| = |\det A_t(x)| \cdot |B_t(x)| \cdot \sin \angle(E_x, F_x)$ and since $B_t(x) = DX_{-t} \mid F_x$ we have $|B_t(x)| \leq C e^{-t\lambda/2}$, hence $|\det A_t(x)|$ grows exponentially fast.

This shows that Λ has a continuous invariant splitting $E \oplus F$ with F uniformly contracting and E area expanding for the backward flow. In a three-dimensional manifold this shows that E is sectionally-expanding for $-X$. From Corollary 1 we have that Λ is a hyperbolic set for $-X$, hence Λ is a hyperbolic set and the proof of Theorem C and Corollary 2 is complete.

7. EXAMPLES

This section is devoted to present some examples.

7.1. Proof of Proposition 1. Here we present examples stated in Proposition 1.

Item (1): Consider a vector field on the plane $\mathbb{R}^2$ with a double saddle homoclinic connection which expands volume and multiply this vector field by a contraction along the vertical direction; see Figure 1. For the construction of the planar vector field, see e.g. [23, Chapter 4, Section 9].

The compact invariant set W , for the vector field X , formed by the pair of homoclinic connections together with the singularity s admits a splitting $E \oplus F$, where E is the vertical direction and F is the plane direction. This splitting is continuous, E is uniformly contracting and F is uniformly area expanding.

Indeed, since we assume that the singularity s expands area, i.e., $|\det DX_t(x)|_{L_x}| \geq c e^{\lambda t}$ for all $t > 0$ and all 2-plane $L_x \subset F_x$, for some fixed constants $c, \lambda > 0$, we can fix a neighborhood U of s such that

$$|\det DX_t(x)|_{L_x}| \geq c e^{\lambda t}, \forall 0 \leq t \leq 1, \forall x \in U$$

and consequently

$$|\det DX_t(x)|_{L_x}| \geq c e^{\lambda t}, \forall t > 0$$

whenever the positive orbit of x is contained in U . Now we observe that there exists $T > 0$ such that for all $x \in W \setminus U$ we have $X_t(x) \in U$ for all $t \in \mathbb{R}$ satisfying $|t| > T$. Hence we can choose

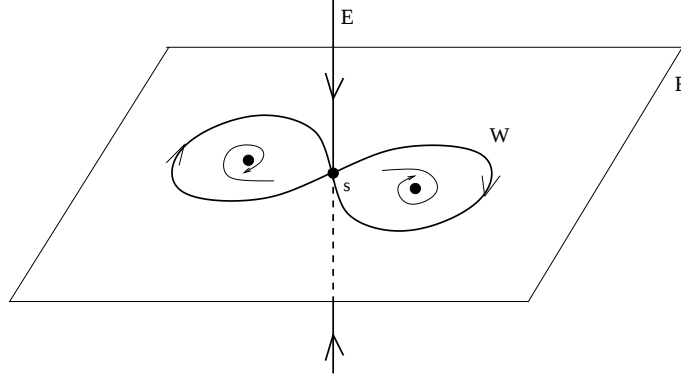


FIGURE 1. A double homoclinic saddle connection expanding area

$k = \inf\{c^{-1}e^{-\lambda t}|\det DX_t(x)|_{L_x}| : x \in W \setminus U, 0 \leq t \leq T\} > 0$ and write $|\det DX_t(x)|_{L_x}| \geq k \cdot ce^{\lambda t}$ for $0 \leq t \leq T$; and for $t > T$

$$|\det DX_t(x)|_{L_x}| = |\det DX_{t-T}(X_T(x))|_{L_{X_T(x)}}| \cdot |\det DX_T(x)|_{L_x}| \geq ce^{\lambda(t-T)} \cdot k \cdot ce^{\lambda T} = kc^2e^{\lambda t}.$$

So we can find a constant $\kappa > 0$ so that $|\det DX_t(x)|_{L_x}| \geq \kappa e^{\lambda t}$ for all $x \in W$ and every $t \geq 0$.

To obtain a *non-dominated* splitting, just choose the contraction rate along E to be weaker than the contraction rate of the singularity s .

Example 3. Considering, in the previous Item (1), Λ as the union of the saddle s with only one homoclinic connection, we have $T_\Lambda M = E \oplus F$ a non-dominated continuous splitting over a compact invariant and isolated set containing a singularity of the flow. Moreover, Λ is the α -limit set of all points of the plane $\mathbb{R}^2 \times \{0\}$ in the region bounded by the curve Λ except the sink; see [23, Chapter 4, Section 9].

Item (2): Consider the flow known as “Bowen example”; see e.g. [29] and Figure 2. We have

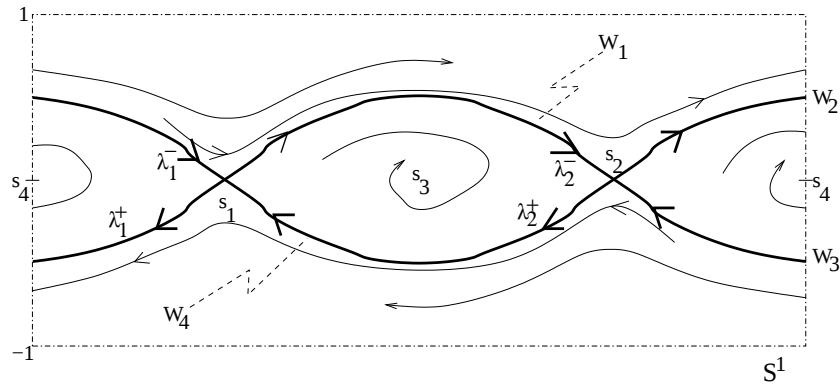


FIGURE 2. A sketch of Bowen's example flow.

chosen to reverse time with respect to the flow studied in [29] so that the heteroclinic connection $W = \{s_1, s_2\} \cup W_1 \cup W_2 \cup W_3 \cup W_4$ is now a repeller. The past orbit under this flow ϕ_t of every point $z \in S^1 \times [-1, 1] = M$ not in W accumulates on either side of the heteroclinic connection, as suggested in the figure, if we impose the condition $\lambda_1^- \lambda_2^- < \lambda_1^+ \lambda_2^+$ on the eigenvalues of the saddle fixed points s_1, s_2 (for more specifics on this see [29] and references therein). In this setting, the saddle singularities s_1, s_2 are area expanding: $|\det D\phi_t(s_i)|$ grows exponentially with $t > 0$, $i = 1, 2$.

We can now embed this system in the 2-sphere putting two sinks at the “north and south poles” of $\mathbb{S}^2$; see Figure 3. Let us denote the corresponding vector field on $\mathbb{S}^2$ by X .

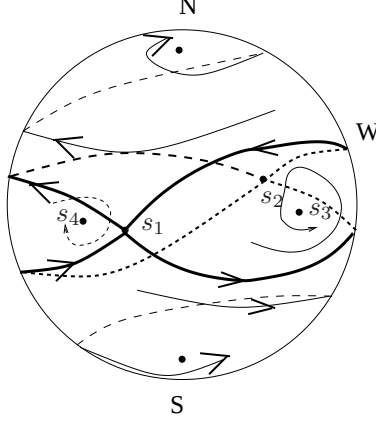


FIGURE 3. The Bowen flow embedded in a sphere

Let Y be a vector field in $\mathbb{S}^1$ corresponding to the “north-south” system: the corresponding flow has only two fixed points N and S , N is a source and S a sink. We can choose the absolute value of the eigenvalues of Y at N, S to be between the absolute value of the eigenvalues of s_1, s_2 .

In this way, for the flow $\{Z_t\}_{t \in \mathbb{R}}$ associated to the vector field $X \times Y$ on $\mathbb{S}^2 \times \mathbb{S}^1$, the singularities $\{s_i \times S\}_{i=1,2}$ of the invariant set $\Lambda := W \times \{S\}$ are hyperbolic saddles, and the splitting $T_\Lambda(\mathbb{S}^2 \times \mathbb{S}^1) = E \oplus F$ given by

$$E_{(p,S)} = \{0\} \times T_S \mathbb{S}^1 \quad \text{and} \quad F_{(p,S)} = T_p \mathbb{S}^2 \times \{0\} \quad \text{for } p \in W,$$

is continuous, DZ_t -invariant, E is uniformly contracted and F is area expanding (by an argument similar to the previous example), but it is not dominated.

Example 3 and items (1) and (2) of Proposition 1 show that Theorem A is false without the dominating condition on the splitting $E \oplus F$ at the set of singularities within Λ .

Item (3): Now we present a suspension flow whose base map has a dominated splitting but the flow does not admit any dominated splitting.

Let $f : \mathbb{T}^4 \times \mathbb{T}^4$ be the diffeomorphism described in [8] which admits a continuous dominated splitting $E^{cs} \oplus E^{cu}$ on $\mathbb{T}^4$, but does not admit any hyperbolic (uniformly contracting or expanding) subbundle. There are hyperbolic fixed points of f satisfying (see Figure 4):

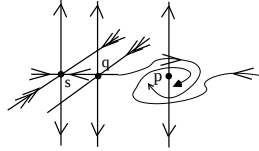


FIGURE 4. Saddles with real and complex eigenvalues.

- $\dim E^u(p) = 2 = \dim E^s(p)$ and there exists no invariant one-dimensional sub-bundle of $E^u(p)$;
- $\dim E^u(\tilde{p}) = 2 = \dim E^s(\tilde{p})$ and there exists no invariant one-dimensional sub-bundle of $E^s(\tilde{p})$;
- $\dim E^s(\tilde{q}) = 3$ and $\dim E^u(q) = 3$.

Hence, the suspension flow of f with constant roof function 1 does not admit any dominated splitting. In fact, the natural invariant splitting $E^{cs} \oplus E^X \oplus E^{cu}$ is the continuous invariant

splitting over $\mathbb{T}^4 \times [0, 1]$ with bundles of least dimension, and is not dominated since at the point p the flow direction $E^X(p)$ dominates the $E^{cs}(p) = E^s(p)$ direction, but at the point q this domination is impossible.

This completes the proof of Proposition 1.

7.2. Proof of Proposition 2. We present first some auxiliary notions.

Let $f : M \rightarrow M$ be a C^1 -diffeomorphism and $\Gamma \subset M$ a compact f -invariant set, that is, $f(\Gamma) = \Gamma$. We say that Γ is *transitive* if there exists $x \in \Gamma$ with dense orbit: the closure of $\{f^n(x) : n \geq 1\}$ equals Γ . The invariant set Γ is *topologically mixing* if, for each pair U, V of non-empty open sets of Γ , there exists $N = N(U, V) \in \mathbb{Z}^+$ such that $U \cap f^n(V) \neq \emptyset$ for all $n > N$.

We say that Γ is an attracting set if there exists a neighborhood U of Γ in M such that $\bigcap_{n \geq 0} f^n(U) = \Gamma$; in this case we say that U is an isolating neighborhood for Γ . An *attractor* is a transitive attracting set and a *repeller* is an attractor for the inverse diffeomorphism f^{-1} . It is well-known that hyperbolic attractors or repellers can be decomposed into finitely many compact subsets which are permuted and each of these pieces is topologically mixing for a power of the original map; see e.g. [25].

Now, we present the example stated in Proposition 2. Consider a hyperbolic attractor (a Plykin attractor, see [24]) Γ , with an isolating neighborhood U , defined on the two-dimensional sphere $\mathbb{S}^2$ for a diffeomorphism $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, with a splitting $T_\Gamma \mathbb{S}^2 = E \oplus F$ satisfying $\|Df|_E\| \leq \lambda_s$ and $\|Df|_F\| \geq \lambda_u$ where $0 < \lambda_s < 1 < \lambda_u$ and $\lambda_s \cdot \lambda_u < 1$.

Now we consider the following diffeomorphism

$$g = f_1 \times f_2 : M \rightarrow M, \quad \text{with} \quad M = \mathbb{S}^2 \times \mathbb{S}^2,$$

where $f_1 := f^2$ and $f_2 := f^{-1}$. We note that $\Gamma_1 := \bigcap_{n \geq 0} f_1^n(U)$ is a hyperbolic attractor with respect to f_1 whose contraction and expansion rates along its hyperbolic splitting $T_{\Gamma_1} \mathbb{S}^2 = E_1 \oplus F_1$ are λ_s^2, λ_u^2 , respectively. Likewise $\Gamma_2 := \bigcap_{n \leq 0} f_2^n(U)$ is a hyperbolic repeller with respect to f_2 whose contraction and expansion rates along its hyperbolic splitting $T_{\Gamma_2} \mathbb{S}^2 = E_2 \oplus F_2$ are $\lambda_u^{-1}, \lambda_s^{-1}$, respectively. This implies in particular that Df_1 contracts area near Γ_1 , that is, $\lambda_s^2 \cdot \lambda_u^2 < 1$ and that Df_2 expands area near Γ_2 , that is, $(\lambda_s \cdot \lambda_u)^{-1} > 1$. Moreover, the set

$$\Gamma := \Gamma_1 \times \Gamma_2$$

is a hyperbolic set for g , whose hyperbolic splitting is given by $(E_1 \times E_2) \oplus (F_1 \times F_2)$.

Now we note, on the one hand, that the splitting $T_\Gamma M = E \oplus F$ with $E = E_1$ and $F = F_1 \oplus E_2 \oplus F_2$ is not dominated, since the contraction rate along E_2 is stronger than the contraction rate along E_1 , but E is uniformly contracted and F is uniformly sectionally-expanded, since the Lyapunov exponents $2 \log \lambda_u, -\log \lambda_u, -\log \lambda_s$ are such that each pair has positive sum.

On the other hand, the decomposition $T_\Gamma M = T_{\Gamma_1} \mathbb{S}^2 \times T_{\Gamma_2} \mathbb{S}^2 = E \oplus F$ where $E = E_1 \oplus F_1$ and $F = E_2 \oplus F_2$ is continuous, since each subbundle is continuous; is also Dg -invariant; but it is not a dominated decomposition. In fact, we have the contraction/expansion rates $0 < \lambda_s^2 < 1 < \lambda_u^2$ associated to E and the corresponding rates $\lambda_u^{-1}, \lambda_s^{-1}$ associated to F , satisfying the relation $\lambda_s^2 < \lambda_u^{-1} < 1 < \lambda_s^{-1} < \lambda_u^2$ (since $0 < \lambda_s \cdot \lambda_u < 1$ and $0 < \lambda_s < 1 < \lambda_u$ imply that $\lambda_s^2 < \lambda_u^{-1}$ and $\lambda_s^{-1} < \lambda_u^2$). Therefore the splitting $E \oplus F$ cannot be dominated. In addition, as noted above, both $Dg^{-1}|_E$ and $Dg|_F$ expand area, so that E is sectionally-contracted and F is sectionally-expanded.

Finally, since both $f_1|_{\Gamma_1}$ and $f_2|_{\Gamma_2}$ are topologically mixing, then $g|_\Gamma$ is transitive; see e.g. [32].

This example shows that there are transitive hyperbolic sets with a non-dominated, although continuous, splitting satisfying the sectional-expansion and uniform contraction or sectional contraction conditions. This completes the proof of Proposition 2.

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(A.A.) UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, INSTITUTO DE MATEMÁTICA, P. O. Box 68530, 21945-970
RIO DE JANEIRO, BRAZIL

E-mail address: `arbieto@im.ufrj.br`

(L.S.) INSTITUTO DE MATEMÁTICA PURA E APLICADA - ESTRADA DONA CASTORINA, 110, JARDIM BOTÂNICO,
22460-320 RIO DE JANEIRO, BRAZIL

E-mail address: `lsalgado@impa.br`