Questions and Answers in General Topology
30 (2012), pp. 1–8

A TOPOLOGICAL STATEMENT AND ITS RELATION TO CERTAIN WEAK CHOICE PRINCIPLES

SAMUEL G. DA SILVA

(Communicated by Yasunao Hattori)

Abstract. In this paper, we investigate how a (topological) statement about accumulation points of subsets of T_1 topological spaces relates to various weak choice principles. Most of the principles under discussion are weak versions (more precisely, restrictions) of the principle of dependent choices (**DC**). For instance, our topological statement (which turns out to be equivalent to the statement "All infinite sets are Dedekind-infinite") implies the restriction of **DC** to lattices in which all principal ideals are finite. We also discuss in which ways several weak choice principles are related to each other, posing a number of questions.

1. Introduction

Throughout this paper, **ZF** is Zermelo-Fraenkel set theory, **AC** is the Axiom of Choice and **ZFC** = **ZF** + **AC**.

Let $\langle \mathbb{P}, \leqslant \rangle$ be a partial order. Defining $<$ in the obvious way, we will say that $\langle \mathbb{P}, < \rangle$ is a *strict partial order*. Equivalently, $\langle \mathbb{P}, < \rangle$ will be said a strict partial order if $<$ is an irreflexive and transitive (hence assymetric) relation on $\mathbb{P}$.

A partial order $\langle \mathbb{P}, \leqslant \rangle$ will be said a *lattice partial order* (or *lattice*) if every pair of elements of $\mathbb{P}$ has a supremum and an infimum, and in this case the corresponding strict partial order $\langle \mathbb{P}, < \rangle$ will be said a *strict lattice partial order*.

A set is said to be *finite* if it is equinumerous to a natural number (and we intend the finite ordinals to be the natural numbers, so if $n > 0 = \emptyset$ then $n = \{0, \ldots, n-1\}$). The set of the natural numbers is denoted by ω – which is also the least limit ordinal. A set is said to be *Dedekind-finite* if it is not equinumerous to a proper subset. "Infinite" means "not finite" and "Dedekind-infinite" means "not

2010 *Mathematics Subject Classification.* Primary 03E25, 06B30; Secondary 54A35, 54D10.

Key words and phrases. Weak choice principles; Principle of dependent choices; Lattice partial orders; T_1 spaces.

This paper was presented at the conference TACL 09 (Topology, Algebra and Categories in Logic - 2009), Amsterdam, The Netherlands. The author thanks FAPESB, Fundacao de Amparo a Pesquisa do Estado da Bahia, for the financial support.

Dedekind-finite". It is well-known that a set is Dedekind-infinite if, and only if, it includes a countably infinite subset. By the Pigeonhole Principle, finite sets are Dedekind-finite (and, therefore, Dedekind-infinite sets are infinite), but the reverse implication depends on the assumption of some choice principle - for instance, the Principle of Dependent Choices.

The Principle of Dependent Choices (**DC**) is a well-known weak choice principle. We may regard **DC** as the fragment of **AC** that allows us to make a countable number of consecutive arbitrary choices ([6]). More precisely:

Principle 1.1 (**DC (Principle of Dependent Choices)**). *Let R be a binary relation over a non-empty set A such that $(\forall x \in A)(\exists y \in A)[xRy]$. Then there is a sequence $\langle x_n : n < \omega \rangle$ of elements of A such that $x_n R x_{n+1}$ for all $n < \omega$.*

Notice that we can restrict our concerns to *irreflexive* relations – if there is a single element $x \in A$ such that xRx, one may merely consider the constant sequence of value x and there is nothing more to be done.

One has **AC** $\Rightarrow$ **DC** $\Rightarrow$ $\mathbf{AC}_\omega$ (where $\mathbf{AC}_\omega$ is the Countable Axiom of Choice, which is the restriction of **AC** to countable families of non-empty sets). None of the implications may be reversed, by results due to Feferman, Jensen and Pincus ([5, Models $\mathcal{M}2$ and $\mathcal{N}21$] and [6, Problem 5.26 and Theorem 8.12]). The following facts are "folklore":

- $\mathbf{AC}_\omega$ is enough to show the equivalence between the notions of infiniteness and Dedekind-infiniteness.
- **DC** is seen by some mathematicians as being "more constructive" than **AC**. Notice that, if $\langle x_n : n < \omega \rangle$ is a sequence given by dependent choices, then it has been possible to choose, in a constructive way, *each one* of the elements of the sequence, because finitary logic allow us to make a finite number of arbitrary choices – what we need **DC** for is to ensure the existence of the sequence itself. In this sense **ZF** + **DC** would be a "first approximation" for constructive mathematics.
- **DC** is not "strong enough" to ensure the existence of non-measurable subsets of $\mathbb{R}$ ([7]).
- **DC** is equivalent to the Baire Theorem for complete metric spaces ([1], [5, Note 28] and [4, Theorem 4.106]).

The notation we use for the Axiom of Countable Choice, $\mathbf{AC}_\omega$, is from [6].

If the reader is familiar with the reference [5], he will be pleased to recall that, in book referred to, **DC** is Form 43 and $\mathbf{AC}_\omega$ is Form 8. $\mathbf{AC}_\omega$ is denoted by **CC**, Countable Choice, in [4]. The statement "All infinite sets are Dedekind-infinite" is Form 9 in [5] and is denoted by **Fin** in [4]. The equivalent form of the last statement, "Every infinite set contains a countable infinite subset", is Form 9G of [5] and is denoted by **DS**, the *Denumerable Subset Axiom*, in some texts (see e.g.

[3]). The implication $\mathbf{AC}_\omega \Rightarrow \mathbf{Fin}$ is also not reversible, by results due to Lévy, Pincus and Jech ([5, Model $\mathcal{N}16$] and [6, Theorem 8.6]).

$\mathbf{AC}_\omega\mathbf{(Fin)}$ (or $\mathbf{CC(Fin)}$), the restriction of the Axiom of Countable Choice to countable families of non-empty finite sets, is Form 10 of [5]. We remark that **Fin** $\Rightarrow \mathbf{AC}_\omega\mathbf{(Fin)}$ ([4, Theorem 2.14]).

CUT denotes the "Countable Union Theorem" – i.e., the statement "The countable union of countable sets is a countable set", Form 31 of [5]. It is easy to see that $\mathbf{AC}_\omega$ implies **CUT**.

CUT(Fin), which is the statement "The countable union of finite sets is a countable set", is equivalent to $\mathbf{AC}_\omega\mathbf{(Fin)}$ ([4, Proposition 3.5]).

The paper is organized as follows. In Section 2 we introduce a certain topological statement, "**(B)**", and we show that it is equivalent to **Fin**. We also present a kind of "more constructive proof" of **DC** $\Rightarrow$ **(B)**; this proof will be used later for comparisons and adaptations. In Section 3 we show that **(B)** implies a restricted version of **DC**. We will be particularly interested in restrictions of **DC** to lattices, and two associated problems are posed in this section. In Section 4 we give a more precise estimate of the consistency strength of the statement **(B)** and we pose a number of questions – some of them being particular instances of the two problems presented in Section 3.

2. Two topological statements

Recall that a topological space is said to be a T_1 space if all the singletons are closed. Equivalently, X is T_1 if, and only if, all finite sets are closed.

Let X be a topological space, $A \subseteq X$ and $x \in X$. We say that x is an *accumulation point* of A if every neighbourhood of x contains points of $A \setminus \{x\}$.

Let **(A)** and **(B)** be the following statements:

(A): Let X be a T_1 topological space, $A \subseteq X$ and $x \in X$. Suppose x is an accumulation point of A. Then every open neighbourhood of x contains infinitely many points of A.

(B): Let X be a T_1 topological space, $A \subseteq X$ and $x \in X$. Suppose x is an accumulation point of A. Then every open neighbourhood of x includes a countably infinite subset of A.

The statement **(A)** is a theorem of **ZF** (with a kind of indirect, "non-constructive" proof, based on the fact that finite subsets of T_1 spaces are always closed), and clearly **(B)** $\Rightarrow$ **(A)** also holds in **ZF**. The statements **(A)** and **(B)** are equivalent in **ZFC**, however this is not true in **ZF** because finiteness and Dedekind-finiteness are not equivalent notions in **ZF**. Indeed, we show now that **(B)** is equivalent to **Fin**.

Proposition 2.1 (**ZF**)**.** **(B)** $\Leftrightarrow$ **Fin**.

Proof. ($\Leftarrow$) Assuming **Fin**, one has **(A)** $\Rightarrow$ **(B)** (since the Dedekind-infinite sets are precisely those including a copy of ω).

($\Rightarrow$) Let X be infinite set. Consider the cofinite topology on X. Under these assumptions, X is a T_1 space without isolated points. But this means that, taking any $x \in X$, the statement "x is an accumulation point of X and X is an open neighbourhood of x" holds – and so we are done. ■

It follows from the preceding proposition (and "folklore") that one has **DC** $\Rightarrow$ **(B)**. In order to produce some comparisons later, we present here an "explicitly more constructive proof" of **DC** $\Rightarrow$ **(B)** – in the sense that will be clear that *each element* of the countably infinite set will be an element of a finite subset obtained after a finite number of steps, and therefore it was selected in a constructive way.

Note 2.2. *An "explicitly more constructive" proof of* **DC** $\Rightarrow$ **(B)**.

Proof. Let x be an accumulation point of $A \subseteq X$ and let U be an open neighbourhood of x. Consider the partial order $\langle [(U \cap A) \setminus \{x\}]^{<\omega}, \subset \rangle$, where $[(U \cap A) \setminus \{x\}]^{<\omega}$ is the family of all finite subsets of $(U \cap A) \setminus \{x\}$ and $\subset$ denotes strict inclusion.

Let $F \in [(U \cap A) \setminus \{x\}]^{<\omega}$. As $U \setminus F$ is an open neighbourhood of x (since X is T_1), one has just to apply the definition of "accumulation point" to get G of the form $G = F \cup \{a\}$ with $a \in (U \cap A) \setminus \{x\}$ such that $G \in [(U \cap A) \setminus \{x\}]^{<\omega}$ and $F \subset G$.

But, what we have just done is to check that the partial order given by $\langle [(U \cap A) \setminus \{x\}]^{<\omega}, \subset \rangle$ fulfils the conditions of **DC**.

Applying **DC**, we may consider a sequence $\langle F_n : n < \omega \rangle$ in this partial order such that $F_n \subset F_{n+1}$ for all $n < \omega$. Define

$$C = \bigcup_{n<\omega} F_n$$

As **DC** $\Rightarrow$ $\mathbf{AC}_\omega$ $\Rightarrow$ **CUT** $\Rightarrow$ **CUT(Fin)**, we have that C is a countably infinite subset of $U \cap A$, as desired[1]. ■

3. Restrictions of DC

In this section, we investigate the relationship between our topological statement **(B)** and three restrictions of **DC**.

Notation 3.1. *Throughout this paper,*

- $\mathbf{DC}_{po}$ *denotes the restriction of* **DC** *to strict partial orders.*
- $\mathbf{DC}_{l}$ *denotes the restriction of* **DC** *to strict lattice partial orders.*

[1]If one tries to construct here a sequence by just relabelling, it will be still needed an equivalent form of **CUT(Fin)**, namely $\mathbf{AC}_\omega$**(Fin)**, in order to fix an enumeration for each one of the finite sets $F_{n+1} \setminus F_n$.

- $\mathbf{DC}_{l,fp}$ *denotes the restriction of* $\mathbf{DC}$ *to strict lattice partial orders in which all principal ideals are finite.*

Notice that $\mathbf{DC}_{l,fp}$ applies to strict lattice partial orders in which every element has only a finite number of predecessors.

Obviously, $\mathbf{DC} \Rightarrow \mathbf{DC}_{po} \Rightarrow \mathbf{DC}_l \Rightarrow \mathbf{DC}_{l,fp}$. Diener ([2]) showed that $\mathbf{DC} \Leftrightarrow \mathbf{DC}_{po}$.

We now proceed to include **(B)** in our discussion. First, we create a suitable topology on a suitable set.

Proposition 3.2 (ZF). *Suppose $\langle X, <\rangle$ is a non-empty strict lattice partial order satisfying*

$$(\forall x \in X)(\exists y \in X)[x < y].$$

Then there is an "one-point end-extension" given by $\langle Y, <^*\rangle = \langle X \cup \{\infty\}, <^*\rangle$ (where $\infty \notin X$ and $<^* = < \cup \{\langle x, \infty\rangle : x \in X\}$) such that Y can be topologized in such a way that ∞ is a non-isolated point of Y, Y is a T_1 space and any set of the form $\uparrow a := \{y \in Y : a \leqslant^* y\}$ for $a \in X$ is an open neighbourhood of ∞.

Proof. Let $\mathcal{S}$ be the family of subsets of Y given by

$$\mathcal{S} = \{\{a\} : a \in X\} \cup \{\uparrow a : a \in X\},$$

and consider Y with the topology generated by $\mathcal{S}$ as a subbase. Such topology on Y is T_1 (and even T_2, because we will prove that it is a T_1 space with only one non-isolated point). Notice that any set of the form $\uparrow a$ for $a \in X$ is an open neighbourhood of ∞.

What is left to prove is that ∞ is a non-isolated point.

For $\{\infty\}$ to be an open subset of Y, one of the following conditions has to hold:
(i) $\{\infty\} \in \mathcal{S}$; or
(ii) there is an non-empty finite subfamily $\mathcal{S}' \subseteq \mathcal{S}$ whose intersection is $\{\infty\}$.

The condition (i) clearly does not hold. And, because of the hypothesis given by $(\forall x \in X)(\exists y \in X)[x < y]$ and from $\langle X, <\rangle$ being a strict lattice partial order, one can easily check that (ii) also does not hold. Indeed: given $x_1, x_2, \ldots, x_n \in X$, fix (with only a *finite* number of arbitrary choices) $y_1, y_2, \ldots, y_n \in X$ such that $x_i < y_i$ for $1 \leqslant i \leqslant n$. If $a = sup\{y_1, y_2, \ldots, y_n\}$, then we have that $\uparrow a$ is a subset of $\bigcap\{\uparrow x_i : 1 \leqslant i \leqslant n\}$. ■

We are in position to present the main theorem of this section.

Theorem 3.3 (ZF). (B) $\Rightarrow \mathbf{DC}_{l,fp}$.

Proof. Let $\langle X, <\rangle$ be a non-empty strict lattice order such that every element has only a finite number of predecessors and also $(\forall x \in X)(\exists y \in X)$ $[x < y]$. Topologize $Y = X \cup \{\infty\}$ as in the preceding proposition. It follows that Y is a T_1 space and ∞ is a non-isolated point , and therefore ∞ is an accumulation point of

$A = X$. Let $a \in X$. Then $\uparrow a$ is an open neighbourhood of ∞, and it follows from the topological statement **(B)** that $\uparrow a$ has a countably infinite subset C, which we may suppose is included in X.

Fix an enumeration of C, say $C = \{y_n : n < \omega\}$. Let $\langle x_n : n < \omega\rangle$ be the sequence of elements of $\uparrow a$ recursively defined by

$$x_n = \begin{cases} y_0 & \text{if } n = 0 \\ sup\{x_{n-1}, y_n\} & \text{if } n > 0 \end{cases}$$

As every element of $\langle X, <\rangle$ has only a finite number of predecessors, the sequence $\langle x_n : n < \omega\rangle$ has infinite image.

By passing to a subsequence if necessary (with the well-ordering of ω avoiding arbitrary choices at all), we may suppose that the x_n are pairwise distincts. But this clearly give us $x_n < x_{n+1}$ for all $n < \omega$, and therefore $\mathbf{DC}_{l,fp}$ holds, as desired. ■

For the rest of the paper, we will be interested on two related problems. The first of them, in view of Proposition 2.1, is also a search for a **DC**-type equivalent for the statement "All infinite sets are D-infinite".

Problem 3.4. *To determine whether there are restricted classes $\mathcal{C}$ of binary relations such that*

$$\mathbf{DC}_{\mathcal{C}} \Leftrightarrow \mathbf{(B)}.$$

Problem 3.5. *To determine whether there are very restricted classes $\mathcal{C}$ of binary relations such that*

$$\mathbf{DC} \Leftrightarrow \mathbf{DC}_{\mathcal{C}}.$$

For both problems, we are particularly interested in classes of lattices. This declared interest suggests what we mean when we say "very restricted classes of binary relations" in Problem 3.5.

Let us present some examples of restricted classes of binary relations whose corresponding restrictions of **DC** are known to be equivalent to **DC** itself. For the rest of this section, we assume the reader is familiar with trees.

As already mentioned, Diener ([2]) showed that $\mathbf{DC} \Leftrightarrow \mathbf{DC}_{po}$. It is also well-known that there are equivalences between **DC** and restrictions of **DC** to restricted classes of partial orders: for instance, **DC** is equivalent to the statement: "Every tree in which every element has at least one successor has a branch" ([5, Form 43H]). In the preceding equivalence, we may specialize the trees by using *discrete rooted trees.*

A discrete (or locally finite) rooted tree is a strict partial order $\langle T, <\rangle$ with least element $r \in T$ (called the root of T) such that the set of the predecessors of any of its elements is a finite chain, and it is easy to check that **DC** is equivalent to its restriction to discrete rooted trees without terminal vertices.

Therefore, a "very restricted class" of binary relations, for us, would be one satisfying restrictions at least as strong as the ones we cited above (partial orders, trees in which every element has at least one successor, discrete rooted trees without terminal vertices), or even stronger restrictions.

In the last section, we will ask if the class $\mathcal{C}$ of lattices would give a positive answer for Problem 3.5.

4. The status of (B) and a number of questions

We end this paper presenting a number of questions. Even without having answers for none of these questions, the author intends to give publicity for them.

Our topological statement **(B)** was shown to be "sandwiched" between **DC** and $\mathbf{DC}_{l,fp}$, meaning

$$\mathbf{DC} \Rightarrow \mathbf{(B)} \Rightarrow \mathbf{DC}_{l,fp}$$

We know that the first implication may not be reversed, since $\mathbf{(B)} \Leftrightarrow \mathbf{Fin}$. Before merely asking about the reversibility second implication, let us give a more precise estimate of the consistency strength of **(B)**.

Theorem 4.1 (**ZF**). $\mathbf{(B)} \Leftrightarrow \mathbf{DC}_{l,fp} + \mathbf{AC}_\omega\mathbf{(Fin)}$.

Proof. ($\Rightarrow$) We have $\mathbf{(B)} \Leftrightarrow \mathbf{Fin}$ (Proposition 2.1) and we also have $\mathbf{Fin} \Rightarrow \mathbf{AC}_\omega\mathbf{(Fin)}$, as already remarked. As **(B)** also implies $\mathbf{DC}_{l,fp}$ (by Theorem 3.3), "($\Rightarrow$)" has been proved.

($\Leftarrow$) For this implication, one has just to consider our "explicitly more constructive proof" of $\mathbf{DC} \Rightarrow \mathbf{(B)}$ (Note 2.2). That proof was made using a strict lattice partial order where all principal ideals are finite – the family of all finite subsets of a given set, ordered by strict inclusion. And, at the end of the argument, what we needed in order to guarantee the countability of C was **CUT(Fin)**, which is equivalent to $\mathbf{AC}_\omega\mathbf{(Fin)}$. So the proof of Note 2.2 is a proof for "($\Leftarrow$)". ■

In view of the preceding theorem, we pose the following question, which also asks about the reversibility of the second implication mentioned in the beginning of this section.

Question 4.2. *Is* $\mathbf{DC}_{l,fp} \Rightarrow \mathbf{AC}_\omega\mathbf{(Fin)}$ *a theorem of* **ZF**?

Notice that if the answer is "yes", then we have solved Problem 3.4, with restricted class $\mathcal{C}$ given by the strict lattice orders with finite principal ideals.

Let us turn to the consistency strength of $\mathbf{DC}_l$. We ask the following questions:

Question 4.3. *Is* $\mathbf{DC}_l \Rightarrow \mathbf{AC}_\omega$ *a theorem of* **ZF**? *Or, at least,* $\mathbf{DC}_l \Rightarrow \mathbf{AC}_\omega\mathbf{(Fin)}$?

Question 4.4. *Is* $\mathbf{DC}_l \Rightarrow \mathbf{CUT}$ *a theorem of* **ZF**?

Again with Theorem 4.1 in mind, we notice that a positive answer for any of the two preceding questions give us a positive answer to the following question:

Question 4.5. *Is* $\mathbf{DC}_l \Rightarrow \mathbf{(B)}$ *a theorem of* $\mathbf{ZF}$*?*

A general problem that one finds when tries to get positive answers for any of the preceding four questions is the following: is it possible, somehow, to work on posets of partial functions within a lattice structure?

The author conjectures that the following question (about strengthening Theorem 3.3) has negative answer.

Question 4.6. *Is* $\mathbf{(B)} \Rightarrow \mathbf{DC}_l$ *a theorem of* $\mathbf{ZF}$*?*

Our guess is that the following question, a particular instance of Problem 3.5, is more likely to have a positive answer.

Question 4.7. *Is it possible to show that* $\mathbf{DC} \Leftrightarrow \mathbf{DC}_l$ *holds in* $\mathbf{ZF}$*?*

It seems more reasonable that, if any equivalences hold, these would be $\mathbf{DC} \Leftrightarrow \mathbf{DC}_l$ and/or $\mathbf{(B)} \Leftrightarrow \mathbf{DC}_{l,fp}$.

Acknowledgements. The author would like to thank Marcel Erné for a number of comments and suggestions (made during and after the conference TACL 09).

The author acknowledges Peter Malcolm Johnson for calling his attention to discreted rooted trees.

The author is also grateful to Ofelia Teresa Alas for presenting sharp observations at earlier stages of this work.

References

[1] Blair, C. E., *The Baire category theorem implies the principle of dependent choices*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25**, 10 (1977), 933-934.

[2] Diener, K. H., *A remark on ascending chain conditions, the countable axiom of choice and the principle of dependent choices*, MLQ Math. Log. Q. **40**, 3 (1994), 415-421.

[3] Erné, M., *Finiteness conditions and distributive laws for Boolean algebras*, MLQ Math. Log. Q. **55**, 6 (2009), 572-586.

[4] Herrlich, H., *Axiom of choice*, volume 1876 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2006, xiv+194.

[5] Howard, P., and Rubin, J. E., *Consequences of the axiom of choice*, volume 59 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1998, viii+432.

[6] Jech, T. J., *The axiom of choice*, volume 75 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1973, xi+202.

[7] Solovay, R. M., *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92**, 1970, 1-56.

Instituto de Matemática, Universidade Federal da Bahia, Campus de Ondina, Av. Adhemar de Barros, S/N, Ondina, CEP 40170-110, Salvador, BA, Brazil

E-mail address: samuel@ufba.br

Received November 22, 2010 and revised July 7, 2011