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(a)-SPACES AND SELECTIVELY (a)-SPACES FROM ALMOST DISJOINT FAMILIES

S. G. DA SILVA

Instituto de Matemática, Universidade Federal da Bahia, Campus de Ondina,
Av. Adhemar de Barros, S/N, Ondina, CEP 40170-110, Salvador, BA, Brazil
e-mail: samuel@ufba.br

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Dedicated to Prof. Ofelia Alas, on the occasion of her 70th birthday

Abstract. We investigate a selective version of property (a) and prove a number of results showing that, under certain set theoretical conditions, (a) spaces and selectively (a) spaces behave in a very similar way, at least for separable spaces. Several results regarding the presence of the referred selective version in spaces from almost disjoint families are established; in particular, we give a combinatorial characterization of such presence. Consistent set theoretical hypotheses implying equivalence between being (a) and being selectively (a) within the referred class are presented, as well as hypotheses implying non-equivalence. We also show that the Continuum Hypothesis is independent of the statement asserting the above mentioned equivalence. The paper finishes by presenting some notes and questions on the role of set theoretical assumptions in the subject.

1. Introduction

In what follows, $\omega = \aleph_0$ denotes the set of all natural numbers (and the least infinite cardinal). $[\omega]^\omega$ and $[\omega]^{<\omega}$ denote, respectively, the family of all infinite subsets of ω and the family of all finite subsets of ω. The first uncountable cardinal is denoted by $\omega_1 = \aleph_1$. For a given set X, $|X|$ denotes the cardinality of X. CH denotes the *Continuum Hypothesis*, which is the statement "$\mathfrak{c} = \aleph_1$", where $\mathfrak{c}$ is the *cardinality of the continuum*, i.e., $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$.

Throughout this paper, we work with a *star covering property* (Matveev's Property (a)) and with a *selective version* of it. The reader may find background information on star covering properties in the papers [5]

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and [11]; for selection principles and topology, we refer to the papers [14] and [9].

For small uncountable cardinals like $\mathfrak{a}$, $\mathfrak{p}$ and $\mathfrak{d}$, see [4].

Property (a) was introduced by Matveev in [10], and its selective version[1] was introduced by Caserta, Di Maio and Kočinac in [3].

Definition 1.1 [10]. A topological space X satisfies *property* (a) (or is said to be an (a)*-space*) if for every open cover $\mathcal{U}$ of X and for every dense set $D \subseteq X$ there is a set $F \subseteq D$ which is closed and discrete in X and such that $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\} = X$.

Definition 1.2 [3]. A topological space X is said to be a *selectively* (a)*-space* if for every sequence $\langle \mathcal{U}_n : n < \omega\rangle$ of open covers and for every dense set $D \subseteq X$ there is a sequence $\langle A_n : n < \omega\rangle$ of subsets of D which are closed and discrete in X and such that $\{\mathrm{St}(A_n,\mathcal{U}_n) : n < \omega\}$ covers X.

It is clear that (a) implies selectively (a). It is also easy to see that several classes of spaces satisfy the selective version of property (a); for instance, T_1, σ-compact spaces are selectively (a). In this paper, we prove a number of results showing that, under certain set theoretical hypothesis, properties (a) and selectively (a) behave in a very similar way, at least for separable spaces. This phenomenom is remarkable, since the importance of property (a) in the literature was established by its relationships with notions as countable compactness, normality and metrizability (see, e.g., [8]); so, it is not surprising that its selective version carries some of these ties. For instance, we will prove that, under CH, every Moore, separable, selectively (a)-space is metrizable. This result is a strengthening of one due to Matveev, obtained for (a)-spaces [10].

It is natural, given a class of topological spaces, to wonder under which conditions these notions are equivalent – or not – when restricted to such class. We consider such question for spaces constructed from *almost disjoint families.*

A set $\mathcal{A} \subseteq [\omega]^\omega$ is said to be an *almost disjoint* (or *a. d.*) *family* if for distinct $X, Y \in \mathcal{A}$ one has $|X \cap Y| < \omega$. For every a. d. family $\mathcal{A}$ one may consider the usual corresponding Ψ*-space*, $\Psi(\mathcal{A})$, whose underlying set is given by $\mathcal{A} \cup \omega$. The points in ω are declared isolated and the basic neighbourhoods of a point $A \in \mathcal{A}$ are given by the sets $\{A\} \cup (A \setminus F)$ for $F \in [\omega]^{<\omega}$. Clearly, ω is a dense set of isolated points, $\mathcal{A}$ is a closed and discrete subset of $\Psi(\mathcal{A})$ and it is easy to check that $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (thus, completely regular) first-countable locally compact separable space. Moreover, it is well-known that any Hausdorff, first countable, locally compact

[1] After the submission of this paper, the author learned that selective versions of star covering properties, as well as other similar notions, are becoming to be known as *star selection principles.*

separable space whose set of non-isolated points is non-empty and discrete turns out to be homeomorphic to a Ψ-space (see [4], p. 154).

Let us describe the organization of this paper. In Section 2 we give a combinatorial characterization of the a. d. families whose corresponding Ψ-spaces satisfy the selective version of property (a). In Section 3 we prove a general result regarding the cardinal functions density and extent of selectively (a)-spaces; as a corollary, one gets that selectively (a) spaces from a. d. families must have size less than $\mathfrak{c}$. In Section 4 we present a number of consistency results. In order to obtain some of these results, we give an absolute, ZFC result relating selectively (a) Ψ-spaces to the small cardinal $\mathfrak{d}$. In Section 5 we present some notes and questions, all of them of set theoretical flavour.

2. Combinatorial characterization of selectively (a) Ψ-spaces

In [18], Szeptycki and Vaughan established the following combinatorial characterization of property (a) for Ψ-spaces: given an almost disjoint family $\mathcal{A}$, the corresponding Ψ-space satisfies property (a) if and only if

$$(\forall f : \mathcal{A} \to \omega)\ (\exists P \subseteqq \omega)\ (\forall A \in \mathcal{A})\big[0 < \big|P \cap \big(A \setminus f(A)\big)\big| < \omega\big].$$

In the following proposition, we give a combinatorial characterization of the selective version of property (a) for Ψ-spaces.

PROPOSITION 2.1. *Let $\mathcal{A} \subseteqq [\omega]^\omega$ be an a. d. family. The corresponding space $\Psi(\mathcal{A})$ is selectively (a) if and only if the following property holds: for every sequence $\langle f_n : n < \omega\rangle$ of functions such that $f_n \in {}^{\mathcal{A}}\omega$ for every $n < \omega$, there is a sequence $\langle P_n : n < \omega\rangle$ of subsets of ω satisfying both of the following clauses:*

(i) $(\forall n < \omega)(\forall A \in \mathcal{A})\big[\,|P_n \cap A| < \omega\big]$,

(ii) $(\forall A \in \mathcal{A})(\exists n < \omega)\big[P_n \cap \big(A \setminus f_n(A)\big) \neq \emptyset\big]$.

PROOF. Let $\mathcal{A}$ be an a. d. family and suppose $\Psi(\mathcal{A})$ is selectively (a). Let $\langle f_n : n < \omega\rangle$ be an arbitrary sequence of functions in ${}^{\mathcal{A}}\omega$. For every $n < \omega$, consider the open cover $\mathcal{U}_n$ given by

$$\mathcal{U}_n = \big\{\{A\} \cup \big(A \setminus f_n(A)\big) : A \in \mathcal{A}\big\} \cup \big\{\{i\} : i < \omega\big\}.$$

Notice that, for every $A \in \mathcal{A}$, $\{A\} \cup \big(A \setminus f_n(A)\big)$ is the only open set in $\mathcal{U}_n$ which contains A.

ω is dense and $\Psi(\mathcal{A})$ is supposed to be selectively (a), and so there is a sequence $\langle P_n : n < \omega\rangle$ of closed discrete subsets of ω such that $\big\{\mathrm{St}\,(P_n, \mathcal{U}_n) : n < \omega\big\}$ cover X. Each P_n has no accumulation points, so clause (i) necessarily holds (notice that if a set $X \in [\omega]^\omega$ has infinite intersection with $A \in \mathcal{A}$, then the point $A \in \Psi(\mathcal{A})$ is an accumulation point of the subset $X \subseteqq \Psi(\mathcal{A})$). Clause (ii) holds because every $A \in \mathcal{A}$ must belong to at least one element of the cover $\{\mathrm{St}\,(P_n,\mathcal{U}_n) : n < \omega\}$, and if $A \in \mathrm{St}\,(P_m,\mathcal{U}_m)$ then the only open set of $\mathcal{U}_m$ that contains A must intersect P_m.

On the other hand, suppose $\mathcal{A}$ is an a. d. family such that the described property on arbitrary sequences of functions from $\mathcal{A}$ into ω holds. We claim that $\Psi(\mathcal{A})$ is selectively (a). Indeed, let $\langle \mathcal{U}_n : n < \omega \rangle$ be an arbitrary sequence of open covers of $\Psi(\mathcal{A})$. For every fixed $n < \omega$ consider a refinement $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\mathcal{V}_n$ has only basic open sets and every point of $\mathcal{A}$ belongs to only one open set of the refinement, say

$$\mathcal{V}_n = \big\{ \{A\} \cup (A \setminus k_{n,A}) \big\} \cup \big\{ \{j\} : j < \omega \big\}.$$

Define now $f_n : \mathcal{A} \to \omega$ by putting $f_n(A) = k_{n,A}$ for every $A \in \mathcal{A}$. By hypothesis, we may consider a sequence $\langle P_n : n < \omega \rangle$ of subsets of ω satisfying clauses (i) and (ii). From (i), every P_n has no accumulation points, and from (ii) we have $\mathcal{A} \subseteqq \bigcup \{\mathrm{St}\,(P_n, \mathcal{V}_n) : n < \omega\}$. For every $n < \omega$, let $A_n = P_n \cup \{n\}$. Then each one of the A_n's is also a closed and discrete subset of ω – and, as ω is a set of isolated points, each A_n is a closed discrete subset of any dense set $D \subseteqq \Psi(\mathcal{A})$. Now we have

$$\Psi(\mathcal{A}) = \bigcup \big\{ \mathrm{St}\,(A_n, \mathcal{V}_n) : n < \omega \big\} \subseteqq \bigcup \big\{ \mathrm{St}\,(A_n,\mathcal{U}_n) : n < \omega \big\} \subseteqq \Psi(\mathcal{A})$$

and therefore $\{\mathrm{St}\,(A_n,\mathcal{U}_n) : n < \omega\}$ covers $\Psi(\mathcal{A})$. As the sequence of open covers $\langle \mathcal{U}_n : n < \omega \rangle$ was arbitrarily chosen, the proof finishes. □

3. On the extent of selectively (a) spaces

Recall that the *extent* of a topological space X, $e(X)$, is the supremum of the cardinalities of all closed discrete subsets of X, provided this is an infinite cardinal, or is $\omega = \aleph_0$ otherwise. The *density* of a topological space X, $d(X)$, is the minimum of the cardinalities of all dense subsets of X, provided this is an infinite cardinal, or is $\omega = \aleph_0$ otherwise.

Jones [7] proved in 1937 the classical and widely known result nowadays referred to as *"Jones' Lemma"*, which in its simplest case states that normal, separable spaces cannot include closed discrete subsets of size $\mathfrak{c}$. In [10], Matveev has established an analogous result for (a)-spaces; we refer to such result as *Matveev's (a)-Jones' Lemma*. Matveev's result was established only for the separable case, but it is straightforward to extend his result to the general case (see [15]).

Matveev's result witnesses that there are covering properties which, in a similar way as normality does, constrain the extent of spaces satisfying them – with such constraints stated in terms of their densities. The reader may find information on this kind of phenomena – classes of topological spaces, defined by covering properties, on which extents are constrained in terms of densities – in the papers [12] and [13].

The separable case of the following result was already remarked, without a proof, in [3]. Here we present a proof for the general case, which may be viewed as a selective version of Matveev's result.

THEOREM 3.1. *If X is a selectively (a)-space, then X cannot include closed and discrete subsets of size not smaller than $2^{d(X)}$.*

PROOF. Let D be a dense subset of X of minimal size (i.e., $|D| = d(X)$) and suppose H is a closed discrete subset of X satisfying $|H| \geqq 2^{d(X)}$. We will show that, under these assumptions, X is not selectively (a).

As $|H| > |D|$ we may assume that H and D are disjoint sets. $\left(2^{d(X)}\right)^{\aleph_0} = 2^{d(X)} \leqq |H|$ and so we are allowed to use H to index the family of all sequences of closed discrete subsets of D; let $\{G_x : x \in H\}$ be such a family (with $G_x = \langle G_{x,n} : n < \omega\rangle$ for every $x \in H$).

For every $n < \omega$ and $x \in H$ let $U_{x,n}$ be the open neighbourhood of x given by $U_{x,n} = X \setminus \big(\big(H \setminus \{x\}\big) \cup G_{x,n}\big)$ and consider the open cover of X given by

$$\mathcal{U}_n = \{X \setminus H\} \cup \{U_{x,n} : x \in H\}.$$

Notice that, for all $x \in H$, we have

(1) $U_{x,n} \cap H = \{x\}$ and (2) $U_{x,n} \cap G_{x,n} = \emptyset$,

and it follows from (1) that for every $x \in H$ the open set $U_{x,n}$ is the only element of $\mathcal{U}_n$ which contains x. Moreover, if we consider the sequence of open covers $\langle \mathcal{U}_n : n < \omega\rangle$ then (2) gives us

(3) $x \notin \bigcup \left\{ \operatorname{St}(G_{x,n}, \mathcal{U}_n)\right\}$.

It follows that such sequence witnesses that X is not selectively (a), for if $\langle A_n : n < \omega\rangle$ is an arbitrary sequence of closed discrete subsets of D we only have to consider the point $z \in H$ such that $A_n = G_{z,n}$ for every $n < \omega$, and therefore $z \in X \setminus \bigcup \left\{ \operatorname{St}(A_n, \mathcal{U}_n)\right\}$, by (3). As the sequence $\langle A_n : n < \omega\rangle$ was arbitrarily chosen, the desired follows. □

Spaces from almost disjoint families are separable, so the following corollary is immediate:

COROLLARY 3.2. *If $\Psi(\mathcal{A})$ is a selectively (a)-space, then $|\mathcal{A}| < \mathfrak{c}$.* □

We are able to give some consequences of the preceding theorem in the realm of *Moore spaces,* which are classical objects of set theoretic topology when it comes to metrizability. The reader may find in [5] the proof of the following result. Recall that the *weight* of X, $w(X)$, is the minimum of the cardinalities of all bases for the topology of X, provided this is an infinite cardinal, or is $\omega = \aleph_0$ otherwise.

LEMMA 3.3 (Lemma 2.2.6 from [5]). *If X is a Moore space such that $w(X)$ does not have countable cofinality, then there is a closed discrete subset D of X such that $|D| = w(X)$.* □

Now, we argue just like Matveev did for (a)-spaces in [10] and establish the following:

THEOREM 3.4. *Under* CH, *separable, Moore, selectively* (a)*-spaces are metrizable.*

PROOF. Let X be a separable, Moore, selectively (a)-space. Assuming CH, the only possibilities for $w(X)$ are $w(X) = \aleph_0$ or $w(X) = \aleph_1$ (because Moore spaces are regular), but the second one is avoided combining the preceding lemma with Theorem 3.1, as $\mathrm{cf}\,(2^{\aleph_0}) > \aleph_0$ by König's theorem. It follows that X is a regular space with a countable base, and therefore it is metrizable. □

We remark that there are in ZFC non-separable, Moore, (a) (thus, selectively (a)) spaces which are not metrizable (see Proposition 3 of [8]).

4. Consistency results

It should be clear that, under CH, both properties under investigation are equivalent for Ψ-spaces; in fact, assuming CH, a space $\Psi(\mathcal{A})$ satisfies any of the two properties if and only if $\mathcal{A}$ is countable. On the one hand, Matveev's (a)-Jones Lemma and its selective version (Theorem 3.1) imply, under CH, countability of $\mathcal{A}$ in case of, respectively, $\Psi(\mathcal{A})$ being (a) or selectively (a). On the other hand, countable a. d. families always correspond to *metrizable* Ψ-spaces, since a countable Ψ-space is a regular space with a countable base. Metrizable spaces are paracompact and paracompactness implies property (a) for T_1 spaces (see [10]) – and so we are done.

In fact, the author proved in [16] – with similar arguments – that normality and countable paracompactness share the same behaviour, meaning that, if one assumes CH and considers the *four* properties (normality, countable paracompactness, property (a) and its selective version), then a space from almost disjoint families satisfies any of them if and only if it is countable. This also justifies our interest in selectively (a) spaces from almost disjoint families, since this property is consistently equivalent in the referred class to properties of undeniable importance.

As probably expected, we may use a form of Martin's Axiom for establishing the consistency of the statement asserting the equivalence (between (a) and selectively (a)) with the negation of CH. More specifically, we will use $\mathrm{MA}_{\sigma\text{-centered}}$. It is well-known that the equality $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$ holds, i.e., $\mathfrak{p}$ is the least cardinal for which Martin's Axiom restricted to σ-centered p.o.'s fails [1]. So, the consistent statement "$\omega_1 < \mathfrak{p} = \mathfrak{c}$" is, in fact, equivalent to $\mathrm{MA}_{\sigma\text{-centered}} + \neg\mathrm{CH}$.

Szeptycki and Vaughan [18] have considered a σ-centered p.o. to prove within ZFC that if $|\mathcal{A}| < \mathfrak{p}$ then $\Psi(\mathcal{A})$ has property (a). In the next proposition we apply such result to show that CH is independent of the statement asserting equivalence between (a) and selectively (a) for Ψ-spaces.

Proposition 4.1. *If $\mathfrak{p} = \mathfrak{c}$, then a Ψ-space satisfies property (a) if and only if satisfies its selective version.*

Proof. Assume $\mathfrak{p} = \mathfrak{c}$ and let $\Psi(\mathcal{A})$ be a selectively (a)-space. By Corollary 3.2, one has $|\mathcal{A}| < \mathfrak{c}$. But then $|\mathcal{A}| < \mathfrak{p}$, and the desired follows from the mentioned result due to Szeptycki and Vaughan. □

As $2^\kappa = \mathfrak{c}$ for every $\kappa < \mathfrak{p}$ (in fact, even for every $\kappa < \mathfrak{t}$, see [4][2]), it follows that $2^{\aleph_0} < 2^{\aleph_1}$ is also independent of the statement asserting the equivalence between our two notions for Ψ-spaces.

Let us turn to the consistency of the statement which asserts non-equivalence between the two notions for Ψ-spaces. Next we give a ZFC result regarding a. d. families whose corresponding Ψ-spaces satisfy the selective version of property (a). The proposition has two parts. In the first part, we show that the small cardinal $\mathfrak{d}$ behaves with respect to selectively (a)-spaces from a. d. families in a similar way as $\mathfrak{p}$ does for (a)-spaces in such class. The part (ii) of the following proposition was already remarked in [3] (without a proof), with the hypothesis of maximality implicitly assumed. The arguments for both parts are straightforward variations of those due to Bonanzinga and Matveev (Proposition 2 of [2]), originally made for a selective property other than the selective version of property (a) (and which we decided not to define in this paper).

Proposition 4.2. *Let $\mathcal{A} \subsetneqq [\omega]^\omega$ be an infinite a. d. family.*

(i) *If $|\mathcal{A}| < \mathfrak{d}$, then $\Psi(\mathcal{A})$ is selectively (a).*

(ii) *Suppose $\mathcal{A}$ is maximal. Then $\Psi(\mathcal{A})$ is selectively (a) if and only if $|\mathcal{A}| < \mathfrak{d}$.*

Proof. Before proving both parts of the proposition, we recall the following well-known fact (see Theorem 3.6 of [4]): $\mathfrak{d}$ may be regarded as the minimal cardinality of a dominating family of functions from ω into ω under the pointwisely defined order[3], meaning that $\mathfrak{d}$ is the minimal κ such that

[2] This comment may become obsolete, given the recent announcement (in Arxiv) that Shelah and Malliaris have finally proved $\mathfrak{p} = \mathfrak{t}$.

[3] The order used in the standard definition of $\mathfrak{d}$ is the mod finite one.

there is a family $\{f_\alpha : \alpha < \kappa\} \subseteqq {}^\omega\omega$ such that for every $f : \omega \to \omega$ there is $\alpha < \kappa$ such that $f(n) \leqq f_\alpha(n)$ for every $n < \omega$.

For the first part, let $\mathcal{A}$ be an a. d. family of size $|\mathcal{A}| < \mathfrak{d}$ and let $\langle \mathcal{U}_n : n < \omega \rangle$ be an arbitrary sequence of open covers of X.

For every $A \in \mathcal{A}$ and $n < \omega$, let $U_{A,n}$ be an open neighbourhood of A which belongs to $\mathcal{U}_n$. Define a family of functions $\mathcal{F} = \{f_A : A \in \mathcal{A}\} \subseteqq {}^\omega\omega$ by putting

$$f_A(n) = \min(U_{A,n} \cap \omega)$$

for every $A \in \mathcal{A}$ and $n < \omega$.

As $|\mathcal{F}| \leqq |\mathcal{A}| < \mathfrak{d}$, then $\mathcal{F}$ *is not* a dominating family in the pointwisely defined order, meaning that there is $f : \omega \to \omega$ satisfying the following property: for every $A \in \mathcal{A}$ there is $m < \omega$ such that $f_A(m) < f(m)$.

Define for every $n < \omega$ the finite set of natural numbers

$$A_n = \big\{ k < \omega : 0 \leqq k \leqq f(n) \big\} \cup \{n\}.$$

As $\Psi(\mathcal{A})$ is a T_1 space, finite sets are closed and discrete, and, as ω is a set of isolated points, $\langle A_n : n < \omega \rangle$ is a sequence of closed discrete subsets of any fixed dense set $D \subseteqq \Psi(\mathcal{A})$. We claim that $\big\{ \mathrm{St}\,(A_n, \mathcal{U}_n) : n < \omega \big\}$ covers $\Psi(\mathcal{A})$. Indeed, if $A \in \mathcal{A}$ and $m < \omega$ satisfies $f_A(m) < f(m)$ then $U_{A,m} \cap A_m \neq \emptyset$ and therefore $A \in \mathrm{St}\,(A_m, \mathcal{U}_m)$. The rest follows from $n \in A_n$ for all $n < \omega$.

For the second part, we show that if a maximal a. d. family $\mathcal{A}$ has size not smaller than $\mathfrak{d}$ then $\Psi(\mathcal{A})$ is not selectively (a). Let $\mathcal{A}$ be as in the preceding phrase and consider $\mathcal{A}' \subseteqq \mathcal{A}$ with $|\mathcal{A}'| = \mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{d}\}$ be a dominating family in the pointwisely defined order and fix an enumeration (i.e. a bijective indexation) $\mathcal{A}' = \{A_\alpha : \alpha < \mathfrak{d}\}$. For every $n < \omega$, consider the open cover $\mathcal{U}_n$ given by

$$\mathcal{U}_n = \Big\{ \{A_\alpha\} \cup \big(A_\alpha \setminus f_\alpha(n)\big) : \alpha < \mathfrak{d} \Big\} \cup \{X \setminus \mathcal{A}'\}$$

and notice that for every $n < \omega$ and $\alpha < \mathfrak{d}$ the set $\{A_\alpha\} \cup \big(A_\alpha \setminus f_\alpha(n)\big)$ is the only element of $\mathcal{U}_n$ containing A_α.

Now, let $\langle P_n : n < \omega \rangle$ be an arbitrary sequence of closed discrete subsets of the dense set ω; as $\mathcal{A}$ is supposed to be a MAD family, each one of the P_n's is, necessarily, a finite set. Define $g : \omega \to \omega$ by putting

$$g(n) = \sup(P_n) + 1$$

for every $n < \omega$. Fix $\xi < \mathfrak{d}$ such that $g(n) \leqq f_\xi(n)$ for every $n < \omega$. Then $A_\xi \notin \bigcup \big\{ \mathrm{St}\,(P_n, \mathcal{U}_n) : n < \omega \big\}$. Indeed, by construction one has $\big(A_\xi \setminus f_\xi(n)\big) \cap P_n = \emptyset$ for every $n < \omega$. As the sequence $\langle P_n : n < \omega \rangle$ was arbitrarily chosen, the sequence of open covers $\langle \mathcal{U}_n : n < \omega \rangle$ and the dense set ω witness that $\Psi(\mathcal{A})$ is not selectively (a). $\square$

The strict inequality $\mathfrak{a} < \mathfrak{d}$ is consistent (see Theorem 5.2 of [4]), and Ψ-spaces from MAD families are known not to be (a)-spaces (apply, e.g., Corollary 2.3 of [15]). So, the following consistency result holds:

Corollary 4.3. *It is consistent that there are selectively* (a) *spaces, constructed from almost disjoint families, which are not* (a)*-spaces.*

Indeed, one has just to consider an infinite MAD family of minimal size in a model of $\mathfrak{a} < \mathfrak{d}$.

We give and end to this section by giving a scolium of the proof given for the first part of Proposition 4.2.

Proposition 4.4. *Let* X *be a* T_1 *separable space with* $|X| < \mathfrak{d}$ *and suppose* X *has the following property*:

($*$) *Any dense subset of* X *has a countable, dense subset.*

Under these assumptions, X *is a selectively* (a) *space.*

Proof. Let $\langle \mathcal{U}_n : n < \omega \rangle$ be an arbitrary sequence of open covers of a T_1 separable space X satisfying ($*$), let Y be any dense subset of X and assume, without loss of generality, $\omega \subseteqq Y$ and ω dense in X. Consider $Z = X \setminus \omega$ and for every $z \in Z$ fix $U_{z,n} \in \mathcal{U}_n$ with $z \in U_{z,n}$. Let $\mathcal{F} = \{f_z : z \in Z\} \subseteqq {}^\omega\omega$ be defined by putting

$$f_z(n) = \min(U_{z,n} \cap \omega)$$

for every $z \in Z$ and $n < \omega$. The proof proceeds by mimicking the one given for first part of Proposition 4.2 in such a way that a sequence $\langle A_n : n < \omega \rangle$ of finite (thus, closed discrete) subsets of $\omega \subseteqq Y$ satisfying $\bigl\{ \mathrm{St}\,(A_n, \mathcal{U}_n) : n < \omega \bigr\} = X$ is obtained. Therefore X is selectively (a), as desired. □

There are several classes of T_1 spaces satisfying the statement ($*$) of the preceding proposition, e.g.: spaces with a countable base, or even first countable separable spaces; hereditarily separable spaces; separable spaces with a dense set of isolated points; and so on. It follows that it is consistent that T_1 spaces satisfying ($*$) and with size less than $\mathfrak{c}$ are all selectively (a).

5. Notes and questions

We already remarked, right after Proposition 4.1, that $2^{\aleph_0} < 2^{\aleph_1}$ is independent of the statement which asserts equivalence between (a) and selective (a) for Ψ-spaces; now we will show that the same happens in the other way around, i.e., we show that such statement is independent of $2^{\aleph_0} < 2^{\aleph_1}$.

Proposition 5.1. *The following statement is consistent with* ZFC + $2^{\aleph_0} < 2^{\aleph_1}$: *"There is a* Ψ*-space which is selectively* (a) *but does not satisfy property* (a)*".*

PROOF. In view of our comments right before and after Corollary 4.3, it is clear that the desired follows from the relative consistency of $\mathfrak{a} < \mathfrak{d}$ with $2^{\aleph_0} < 2^{\aleph_1}$. The standard way to give the consistency of $\mathfrak{a} < \mathfrak{d}$ is to consider, in a model M of CH, a cardinal $\kappa > \aleph_1$ satisfying $\kappa^{\aleph_0} = \kappa$ and add κ many Cohen reals; in the extension one has $\mathfrak{a} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c} = \kappa$. Applying such construction in a model M of GCH for $\kappa = \aleph_{\omega_1}$, we get also $2^{\aleph_0} < 2^{\aleph_1}$ in the extension – because if $2^{\aleph_0} = 2^{\aleph_1}$ holds in the extension then $\aleph_1 = \omega_1 = \mathrm{cf}\,(\aleph_{\omega_1}) = \mathrm{cf}\,(2^{\aleph_1})$, but this contradicts König's theorem. □

An open question related to $2^{\aleph_0} < 2^{\aleph_1}$ is the following (Question 3.1 of [15]): it is unknown whether $2^{\aleph_0} < 2^{\aleph_1}$ alone suffices to avoid the existence of separable (a)-spaces with uncountable closed discrete subsets. In a more general setting, we do not know if considering an (a)-space X and $\kappa = d(X)$ one necessarily has $e(X) \leqq d(X)$ under $2^{\kappa} < 2^{\kappa^+}$. We proposed such question in [15] motived by natural comparisons between Jones' Lemma for normal spaces and Matveev's (a)-Jones' Lemma. Indeed, for normal spaces the set theoretical assumption given by $2^{d(X)} < 2^{d(X)^+}$ alone implies $e(X) \leqq d(X)$ – as a corollary of the proof of Jones' Lemma, which is based on a direct construction of an injective function from the power set of a closed discrete subset of a normal space into the power set of a dense subset of the considered space.

We present the following proposition just to point out that $2^{\aleph_0} < 2^{\aleph_1}$ does not avoid the existence of separable, selectively (a) spaces with uncountable extent.

PROPOSITION 5.2. *The following statement is consistent with* ZFC + $2^{\aleph_0} < 2^{\aleph_1}$: *"There is a separable, selectively* (a)*-space with an uncountable closed discrete subset."*

PROOF. By first part of Proposition 4.2, it is clear that it suffices to get a model where both inequalities $2^{\aleph_0} < 2^{\aleph_1}$ and $\aleph_1 < \mathfrak{d}$ hold: in such a model, any almost disjoint family of size $\aleph_1$ will give us the desired consistency, since Ψ-spaces are separable and an almost disjoint family is always a closed discrete subset of its corresponding Ψ-space. But the desired inequalities are already satisfied by the model of the preceding proposition. □

We point out that the relative consistency with respect to ZFC + $2^{\aleph_0} < 2^{\aleph_1}$ of "there is an uncountable a. d. family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ satisfies property (a)" is – maybe surprisingly – related to large cardinals (because there is a relationship between uncountable a. d. families whose corresponding Ψ-space satisfies property (a) and the so-called *small dominating families* in ${}^{\omega_1}\omega$, see [15], and the hypothesis of existence of such families – when combined with the set theoretical assumptions "$2^{\aleph_0} < 2^{\aleph_1}$" and "$2^{\aleph_0}$ is regular" – is related to inner models with measurable cardinals, by results due to Jech and Prikry [6]). It is not clear to the author whether uncountable selectively

(a)-spaces are related to small dominating families; despite the statement of the last proposition, notice that $2^{\aleph_0}$ is not regular in the model exhibited for the consistency of both preceding propositions.

Let us turn to other direction. While it is a ZFC result that $\Psi(\mathcal{A})$ satisfies property (a) whenever $|\mathcal{A}| < \mathfrak{p}$, the existence of an almost disjoint family of size $\mathfrak{p}$ such that $\Psi(\mathcal{A})$ is an (a)-space is shown to be consistent in [18]. In view of Proposition 4.2 it is very natural to ask the following question.

QUESTION 5.3. *Is it consistent that there is an a. d. family $\mathcal{A}$ of size $\mathfrak{d}$ such that $\Psi(\mathcal{A})$ is selectively (a)?*

Of course, a consistent example for a positive answer to the preceding question only can be given by a non-maximal a. d. family, again in view of Proposition 4.2. We also remark that the analogous question for property (a) was already posed (Question 4.3 of [16]) and it remains unanswered.

In fact, there are at least two more questions on Ψ-spaces – focusing if, and how, the presence of property (a) in such class of spaces is related to normality and countable paracompactness – which are still open for many years. One of them is due to Szeptycki [17] and essentialy asks whether normal spaces from almost disjoint families must have property (a)[4]. The other one, due to the author, is the one obtained by replacing, in Szeptycki's question, normality with countable paracompactness [16].

Here we present the selective versions of such questions. Positive answers to the following questions may lead us to positive answers to the referred, long time open questions previously posed for property (a); and negative answers for these questions give negative answers for those ones. In both cases, it seems interesting to consider them.

QUESTION 5.4. *If $\Psi(\mathcal{A})$ is normal, is it a selectively (a)-space?*

QUESTION 5.5. *If $\Psi(\mathcal{A})$ is countably paracompact, is it a selectively (a)-space?*

Every normal Ψ-space is countably paracompact (see [16]), so, considering the last two questions, a positive answer to the latter provides a positive answer to the former.

To finish, notice that all consistency results of this paper were given in terms of small cardinals. Is there a way to describe precisely the possibilities of equivalence between our two properties, when restricted to Ψ-spaces, by using such cardinals?

[4] In fact, Szeptycki's question goes as follows: given an a. d. family $\mathcal{A}$ such that the corresponding Ψ-space is normal, is such a family necessarily *soft* (meaning that there is an infinite $P \subseteq \omega$ such that $0 < |P \cap A| < \aleph_0$ for every $A \in \mathcal{A}$)? This relates directly to property (a) because an a. d. family $\mathcal{A}$ corresponds to a Ψ-space with property (a) if and only if every finite modification of $\mathcal{A}$ is soft. It is worthwhile mentioning that the answer is "yes" in the case of $|\mathcal{A}| < \mathfrak{d}$, by Theorem 3 of [17].

Problem 5.6. Find a statement φ, if any, enunciated in terms of small cardinals, such that (a) and selectively (a) are equivalent properties for Ψ-spaces if and only if φ holds.

We hope that the combinatorial characterization given in Proposition 2.1 will be useful on the discussion of the presented questions and problem.

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